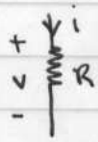
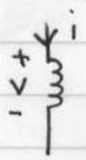
Models of Linear Systems

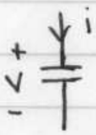
Electrical Circuits



$V = Ri$ Resistor

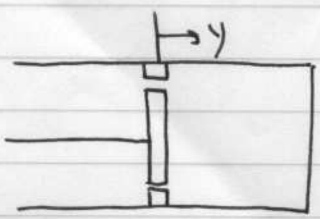


$V = L\frac{\partial i}{\partial t}$ Inductor

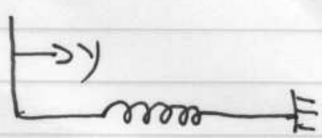


$i = C\frac{\partial v}{\partial t}$ and $V(t) = \frac{1}{C}\int_0^t i(t)\,\partial t$ Capacitor

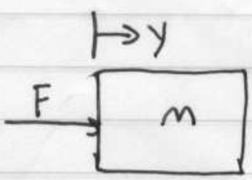
Mechanical Translational Systems



$f = b\dot{y}$ Damper



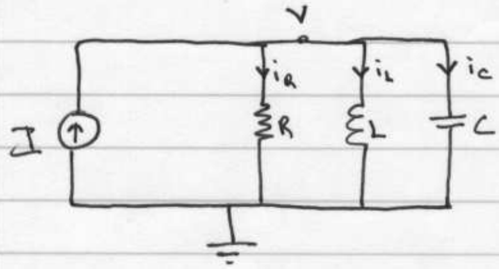
$f = ky$ Spring



$F = m\frac{\partial^2 y}{\partial t^2}$ mass

ex 1)

Electrical



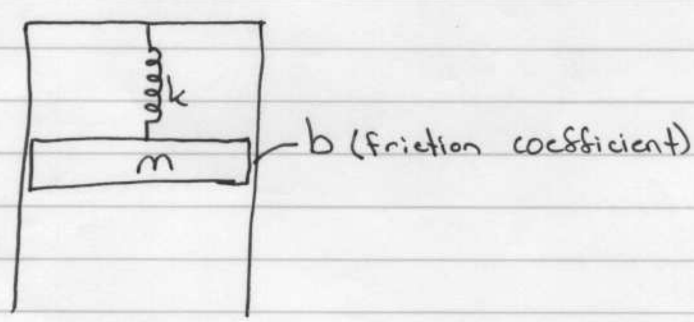
$$I = i_R + i_L + i_C$$

$$i_R = \frac{V}{R} \qquad i_L = \frac{1}{L}\int V \partial t \qquad i_C = C\frac{\partial V}{\partial t}$$

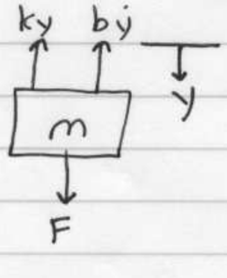
$$I = \frac{V}{R} + \frac{1}{L}\int V \partial t + C\frac{\partial V}{\partial t}$$

$$I(t) = \left(\frac{1}{R} + \frac{1}{L}\int (\,)\partial t + C\frac{\partial (\,)}{\partial t} \right) V$$

Mechanical



Free Body Diagram



$$m\frac{\partial^2 y}{\partial t^2} = F - ky - b\dot{y}$$

$$F = m\frac{\partial^2 y}{\partial t^2} + b\frac{\partial y}{\partial t} + ky$$

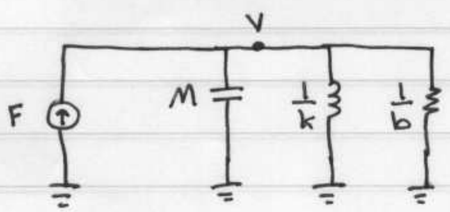
make $v = \frac{\partial y}{\partial t}$

$$F = M\frac{\partial v}{\partial t} + bv + k\int v\,\partial t$$

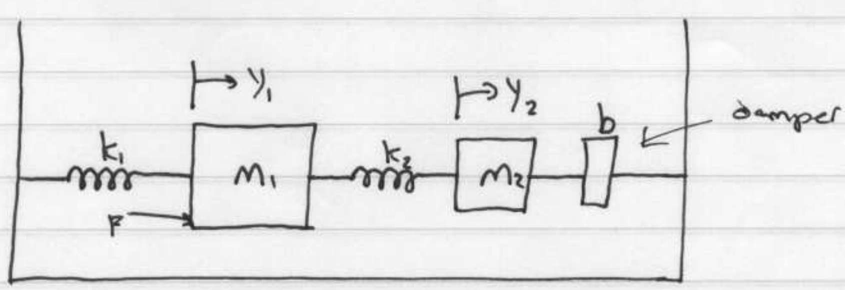
remember

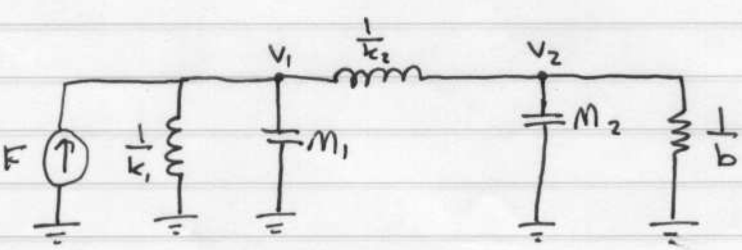
$$I(t) = C\frac{\partial v}{\partial t} + \frac{v}{R} + \frac{1}{L}\int v\,\partial t$$

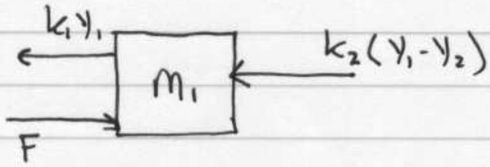
So,



ex2)

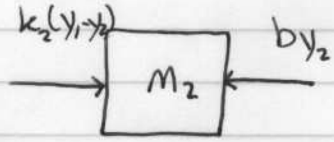




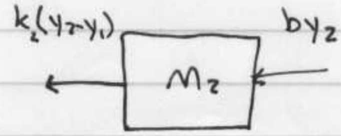


$$m_1 \frac{\partial^2 y_1}{\partial t^2} = F - k_1 y_1 - k_2 (y_1 - y_2)$$

$$F = m_1 \frac{\partial^2 y_1}{\partial t^2} + (k_1 + k_2) y_1 - k_2 y_2$$



or



$$m_2 \frac{\partial^2 y_2}{\partial t^2} = - b \frac{\partial y_2}{\partial t} - k_2 (y_2 - y_1)$$

$$m_2 \frac{\partial^2 y_2}{\partial t^2} + b \frac{\partial y_2}{\partial t} + k_2 y_2 - k_2 y_1 = 0$$

$$\begin{bmatrix} m_1 \frac{\partial^2}{\partial t^2} + (k_1 + k_2) & -k_2 \\ -k_2 & m_2 \frac{\partial^2}{\partial t^2} + b\frac{\partial}{\partial t} + k_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} F \\ 0 \end{bmatrix}$$

Laplace Transform

$$F(s) = \int_{0^-}^{\infty} f(t) e^{-st} \partial t$$

ex) Take $f(t) = u(t)$ $\qquad u(t) = \begin{cases} 1 & 0 \leq t \\ 0 & t < 0 \end{cases}$

$$U(s) = \int_{0^-}^{\infty} e^{-st} \partial t$$

$$= \frac{-e^{-st}}{s} \Big|_0^{\infty} = \frac{-e^{-s\infty} - 1}{s}$$

if $s = \alpha + jw$

$$e^{-st} = e^{-\alpha t} e^{-jwt}$$

$$U(s) = \frac{-\cancelto{0}{e^{-s\infty}} - 1}{s} = \frac{1}{s} \; ; \; 0 < Re\{s\} \qquad (\alpha > 0)$$

$g(t) = f'(t)$

$$G(s) = \int_{0^-}^{\infty} g(t) e^{-st} \partial t = \int f'(t) e^{-st} \partial t \qquad f' = \frac{\partial f}{\partial t}$$

$$= \int e^{-st} \frac{\partial f}{\partial t} \partial t = \int e^{-st} \partial f \qquad \int u\,dv = uv - \int v\,du$$

$u = e^{-st} \quad dv = -se^{-st}$

$v = f$

$$= e^{-st} f \Big|_0^{\infty} - s \int e^{-st} f \partial t$$

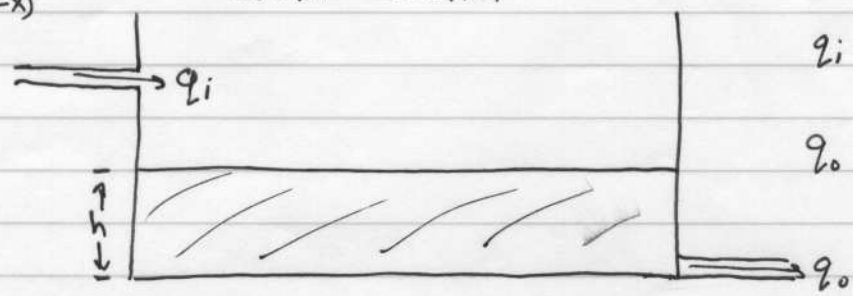
$$G(s) = e^{-st} f(t) \Big|_0^{\infty} + s \int_0^{\infty} f(t) e^{-st} \partial t$$

$$= e^{-s\infty} f(\infty) - e^{-s\emptyset} f(0) + s F(s)$$

$$G(s) = sF(s) - f(0)$$

$\mathcal{L}(f'(t)) = S\mathcal{L}(f) - f(0)$

ex) Water Control



q_i = Volume flow rate for water in

q_o = Volume flow rate for water out

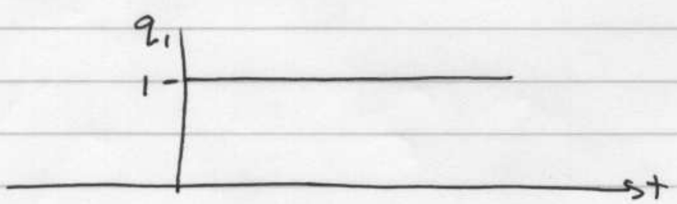
$q_i - q_o = \frac{\partial V}{\partial t}$ Rate of chang of volume w.r.t. time

$= A \frac{\partial h}{\partial t}$

area

assume $q_o = kh$ k is a constant.

$q_i = kh + A\frac{\partial h}{\partial t}$



$Q_1(s) = k H(s) + A(S H(s) - h(0))$

$Q_1(s) = (k + AS) H(s) - A h(0)$

Transfer function $\begin{cases} h(0) = 0 \\ \text{ratio } \frac{\text{output}}{\text{input}}. \end{cases}$

$$\frac{H(s)}{Q_1(s)} = \frac{1}{k + As}$$ Transfer function

Step input $q_1(t) = u(t)$

$$Q_1(s) = \frac{1}{s}$$

$$\frac{1/A}{k/A + s}$$

$$H(s) = \frac{1}{s} \cdot \frac{1}{k + As} = \frac{A}{s} + \frac{B}{s + k/A}$$

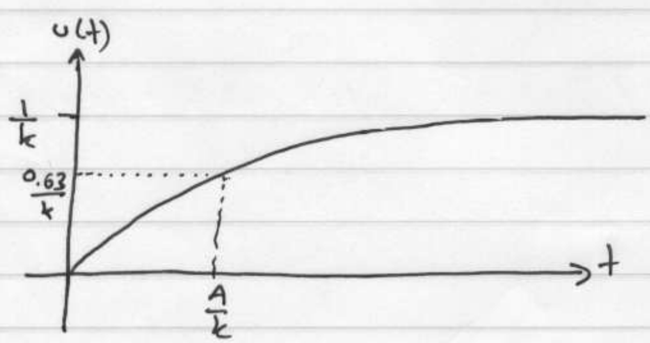
$$A = \frac{1/A}{k/A} = \frac{1}{k}$$

$$B = (s + k/A)\, H(s) \Big|_{s = -k/A} = \frac{1/A}{-k/A} = -\frac{1}{k}$$

$$H(s) = 1/k \left(1 - \frac{1}{s + k/A}\right)$$

$$h(t) = 1/k\, u(t) - 1/k\, u(t) e^{-\frac{k}{A} t}$$

$$= 1/k\, u(t) \left(1 - e^{-\frac{k}{A} t}\right)$$



Final Value Theorem

$$f(t) \xrightarrow{LT} F(s)$$

f.v.t : $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$

subject to $f(t) < \infty$

$$H(s) = \frac{1}{s} \cdot \frac{1}{s+k}$$

$$\lim_{t \to \infty} h(t) = \lim_{s \to 0} sH(s) = \lim \frac{\cancel{s}}{\cancel{s}(s+k)} = \frac{1}{k}$$

Final Value Theorem

If $\lim_{t\to\infty} f(t)$ exists or if $F(s)$ has all of its poles in $Re\{s\} < 0$ (LHP), except possibly one pole at $s=0$ then

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

$$\mathcal{L}(f'(t)) = sF(s) - f(0)$$

$$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

$$\int_0^\infty f'(t)\,dt = \lim_{s\to 0} sF(s) - f(0)$$

$$f'(\infty) - \cancel{f'(0)} = \lim_{s\to 0} sF(s) - \cancel{f(0)}$$

$$f(\infty) = \lim_{s\to 0} sF(s)$$

example:

$$F(s) = \frac{1}{s-1}$$

$$f(t) = e^t$$

Can not equate

The final value theorem can not be used as the system has a pole in the right hand plane.

$$\mathcal{L} f'(t) = sF(s) - f(0)$$

$$\begin{aligned}\mathcal{L} f''(t) &= s\mathcal{L} f'(t) - f'(0)\\ &= s(sF(s) - f(0)) - f'(0)\\ &= s^2 F(s) - sf(0) - f'(0)\end{aligned}$$

$$\mathcal{L}\, f^n(t) = S^n F(s) - s^{n-1} f^0(0) - s^{n-2} f^1(0) \ldots - f^{n-1}(0)$$

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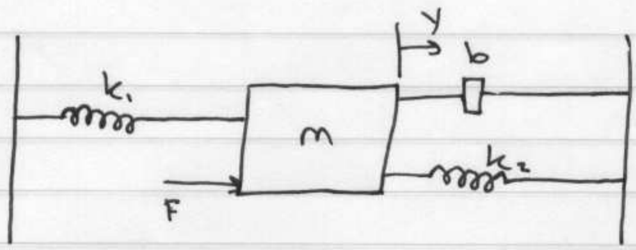
example



Find $\dfrac{Y(s)}{F(s)}$ (transfer function)

$$F - k_1 y - k_2 y - b\dot{y} = m\frac{\partial^2 y}{\partial t^2} \qquad \dot{y} = \frac{\partial y}{\partial t}$$

$$F = m\frac{\partial^2 y}{\partial t^2} + b\frac{\partial y}{\partial t} + (k_1+k_2)y$$

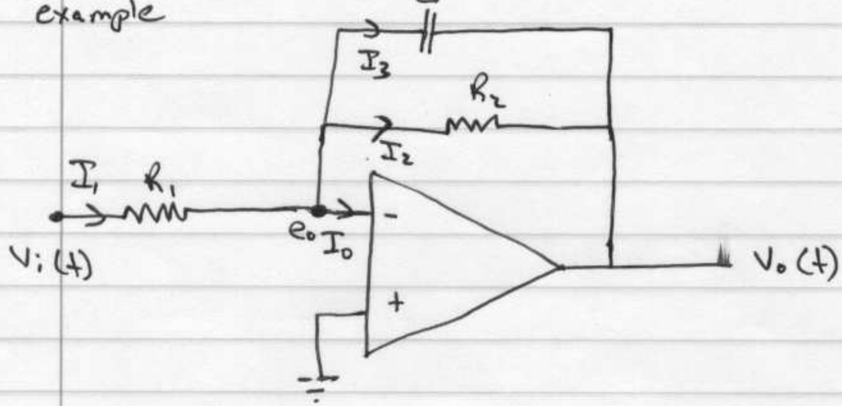
$$F(s) = m(s^2 Y(s) - s\,\cancel{y(0)} - \cancel{y'(0)}) + b(sY(s) - \cancel{y(0)}) + (k_1+k_2)Y(s)$$

To find T.F. set derivatives to zero

$$F(s) = Ms^2 + bs + (k_1+k_2)Y(s)$$

$$\frac{Y(s)}{F(s)} = \frac{1}{M(s^2) + b(s) + (k_1+k_2)}$$

example



• no current flows into op amp.

$I_0 = 0$
$e_0 = 0$

$$I_1 = \frac{V_1 - \cancel{e_0}^{\,0}}{R} \qquad I_2 = \frac{\cancel{e_0}^{\,0} - V_0}{R_2} \qquad I_3 = \frac{c\,\partial(0 - V_0)}{\partial t} = -\frac{c\,\partial V_0}{\partial t}$$

$$I_1 = I_2 + I_3$$

$$\frac{V_1}{R_1} = \frac{-V_0}{R_2} - \frac{c\,\partial V_0}{\partial t}$$

Laplace $\quad \dfrac{V_1}{R_1} = -V_0\left(\dfrac{1}{R_2} + cs\right)$

$$\frac{V_0}{V_1} = \frac{-1/R_1}{1/R_2 + cs} = \frac{-R_2/R_1}{R_2 cs + 1}$$

Take $V_1 = \dfrac{1}{s}$; Find $V_0(\infty)$

First use F.V.T.

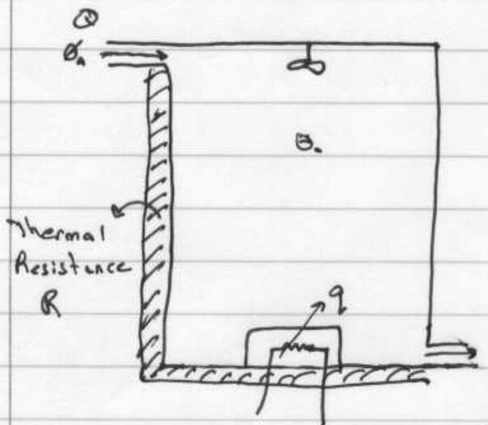
$$V_0(s) = V_1\left(\frac{-R_2/R_1}{R_2 cs + 1}\right)$$

$$V_0(\infty) = \lim_{s\to 0} \frac{\cancel{s}}{\cancel{s}}\left(\frac{-R_2/R_1}{R_2 sc + 1}\right) = \frac{-R_2}{R_1} \Rightarrow \frac{k}{\tau s + 1}$$

k → Steady state gain

τ → time constant

Thermal Heating System



$S \rightarrow$ specific heat
$q \rightarrow$ rate of heat flow
$C_t \rightarrow$ thermal capacitance

Heat going out $\rightarrow QS\theta_o - QS\theta_a = QS(\theta_o - \theta_a)$
Heat throug walls $\rightarrow \dfrac{\theta_o - \theta_a}{R}$

$$q - QS(\theta_o - \theta_a) - \frac{\theta_o - \theta_a}{R} = C_t \frac{\partial \theta_o}{\partial t}$$ energy balance

$\theta_o - \theta_a = \theta$ assume θ_a is a constant

$$q - QS\theta - \frac{\theta}{R} = C_t \frac{\partial \theta}{\partial t}$$

$$q = C_t \frac{\partial \theta}{\partial t} + \left(\frac{1}{R} + QS\right)\theta$$

$q(t) \xrightarrow{L} \hat{q}(s)$
$\theta(t) \rightarrow \theta(s)$

$$\hat{q}(s) = \left(C_t \cdot S + \frac{1}{R} + QS\right)\theta(s)$$

$$\frac{\theta(s)}{\hat{q}(s)} = \frac{1}{C_t S + \frac{1}{R} + QS}$$

$$\frac{d^n y}{dt^n} + q_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots + q_0 y = P_{n-1}\frac{d^{n-1} r}{dr^{n-1}} + P_{n-2}\frac{d^{n-2} r}{dr^{n-2}} + \dots + P_0 r$$

$$S^n Y(s) + q_{n-1}S^{n-1}Y(s) + \dots + q_0 Y(s) = P_{n-1}S^{n-1}R(s) + P_{n-2}S^{n-2}R(s) + \dots + P_0 R(s)$$

$$(S^n + q_{n-1}S^{n-1} + \dots + q_0)Y(s) = (P_{n-1}S^{n-1} + P_{n-2}S^{n-2} + \dots + P_0)R(s)$$

$$\frac{Y(s)}{R(s)} = \frac{P_{n-1}S^{n-1} + \dots + P_0}{S^n + q_{n-1}S^{n-1} + \dots + q_0} = G(s) \text{ transfer function}$$

$$Y(s) = G(s)R(s)$$

The poles are the roots of the denominator.

The zeroes are the roots of the numerator.

Find $Y(\infty)$ when $r(t) = u(t)$

$$Y(\infty) = \lim_{S\to 0} S\,Y(s) = \lim_{S\to 0} S\left(\frac{P_{n-1}S^{n-1} + \dots + P_0}{S^{n-1} + q_{n-1}S^{n-1} + \dots + q_0}\cdot\frac{1}{S}\right)$$

$$Y(\infty) = \frac{P_0}{q_0} = G(0) \quad \text{(DC gain of the system)}$$

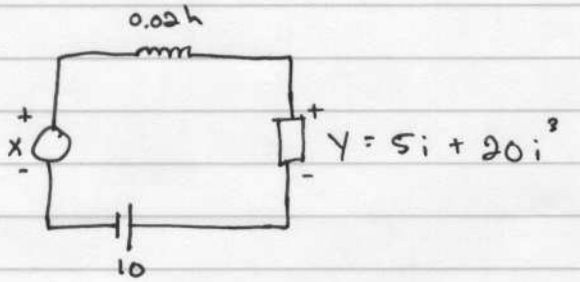
Linearization $\quad Y = f(x) \qquad Y = Y_0 + \Delta Y$
$\qquad\qquad\qquad\qquad\qquad\qquad\quad X = X_0 + \Delta X$

use the taylor series

$$Y_0 + \Delta Y = f(X_0 + \Delta X) = f(X_0) + f'(X_0)\Delta X + \dots$$

$$\boxed{\Delta Y = f'(X_0)\Delta X}$$

<u>example</u>



$i_0 = 0.1\ A$

Find $\frac{\Delta Y}{\Delta X}$

$$X - 10 = (0.02)\frac{\partial i}{\partial t} + 5i + 20i^3 \qquad i_0 = 0.1$$

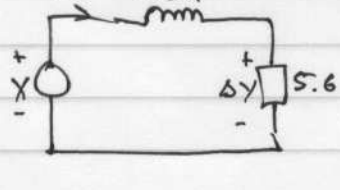
$$X = (0.02)\cancel{\frac{\partial i}{\partial t}}^{0} + 5(0.1) + 20(0.1)^3 + 10 \qquad \text{at steady state, derivatives} = 0$$

$$X = 0.5 + 0.02 + 10$$
$$X = 10.52$$

linearize

$$\Delta Y = (5i + 20i^3)\Big|_{i=0.1}^{\partial i}$$
$$= (5 + 60i^2)\Big|_{i=0.1}^{\partial i}$$

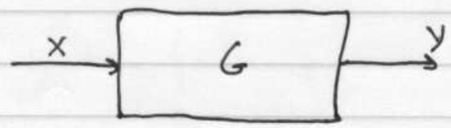
$$\Delta Y = 5.6\,\Delta i$$



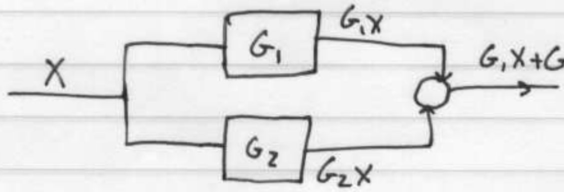
$$\frac{\Delta Y}{\partial X} = \frac{\Delta Y}{\Delta i}\,\frac{\Delta i}{\Delta X} = 5.6\,\frac{1}{0.02s + 5.6}$$

$$\frac{\Delta Y}{\Delta X} = \frac{1}{0.00375s + 1}$$

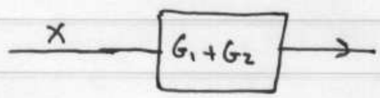
Block Diagram Reduction

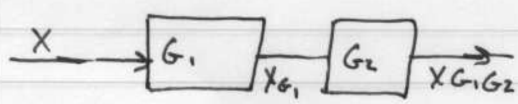


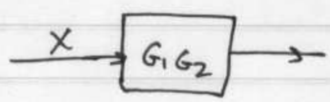
$$G = \frac{Y}{X} \quad ; \quad Y = GX$$

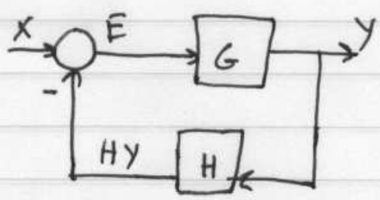


$$G_1X + G_2X = X(G_1+G_2)$$







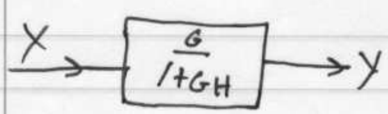


$$E = X - HY$$

$$Y = GE = G(X - HY) = GX - GHY$$

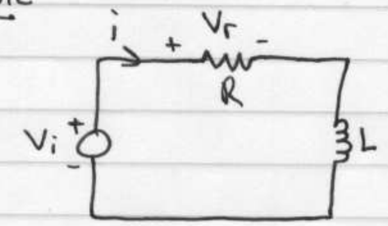
$$Y + GHY = GX$$

$$Y = \frac{GX}{1+GH}$$



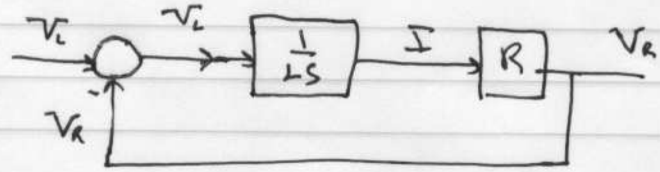
negative feedback $\Rightarrow \dfrac{G}{1+GH}$

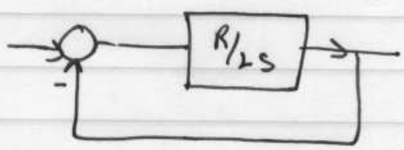
example



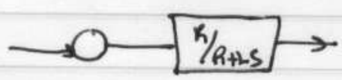
$$V_L = L\frac{di}{dt}$$

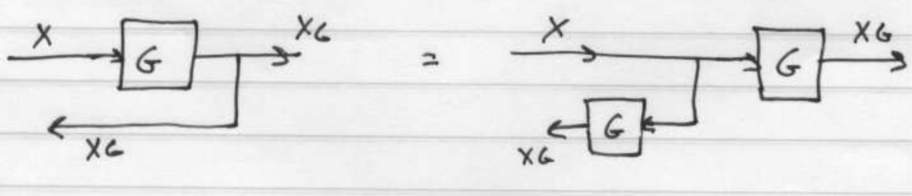
$$V_L = LsI(s)$$

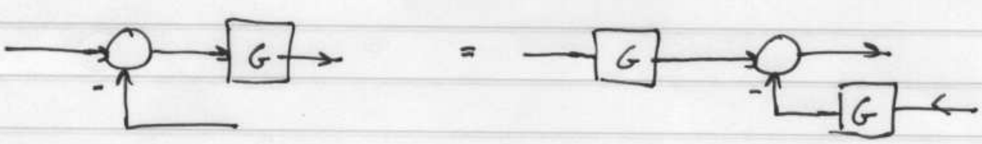




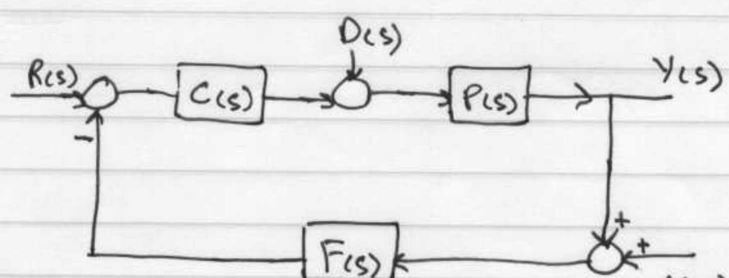
$$\frac{R/Ls}{1+R/Ls} \Rightarrow \frac{R}{R+Ls} \quad (H=1)$$







Example



EE 481
Sept 20/04

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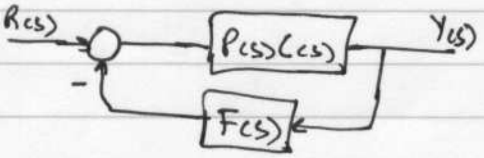
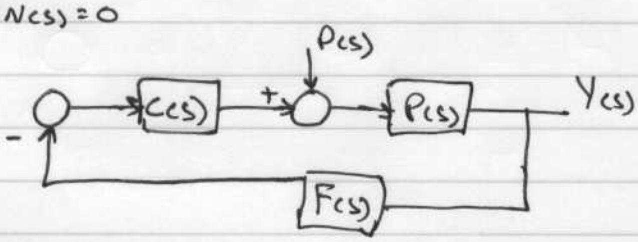
Simplify by using superposition.

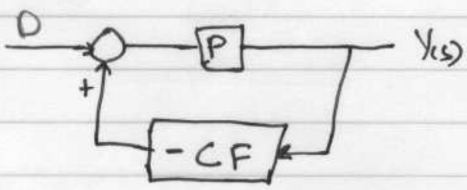
a) $D(s) = N(s) = 0$



$$Y_R(s) = \frac{P.C}{1+P.C.F} \cdot R$$

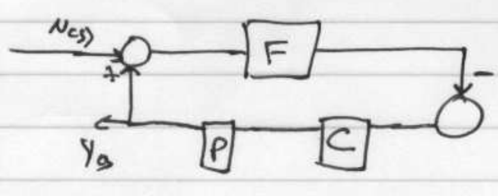
b) $R(s) = N(s) = 0$



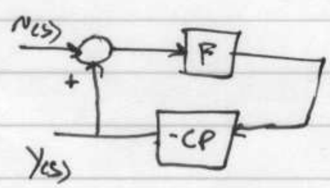


$$Y_D(s) = \frac{P}{1-(-CF)P} \cdot D = \frac{P}{1+PCF} \cdot D$$

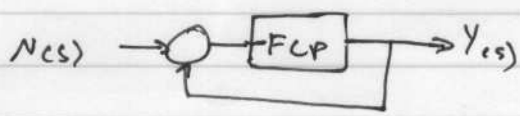
c) $R(s) = D(s) = 0$



$$Y_N(s) = \frac{-FCPN}{1+FCP}$$



=>

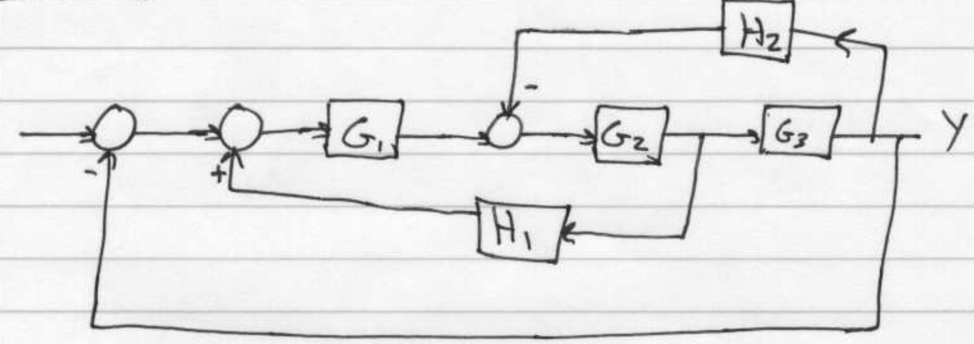


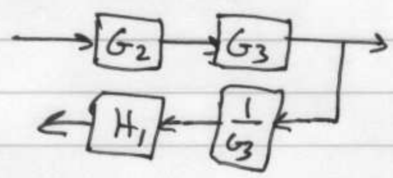
$$Y(s) = Y_R(s) + Y_D(s) + Y_N(s)$$

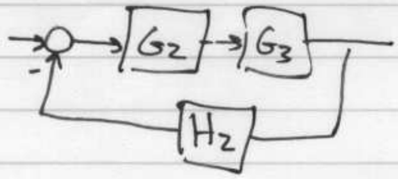
$$= \frac{PC}{1+PCF} \cdot R + \frac{P}{1+PCF} \cdot D + \frac{-FCP}{1+FCP} \cdot N$$

Notice all denominators are the same

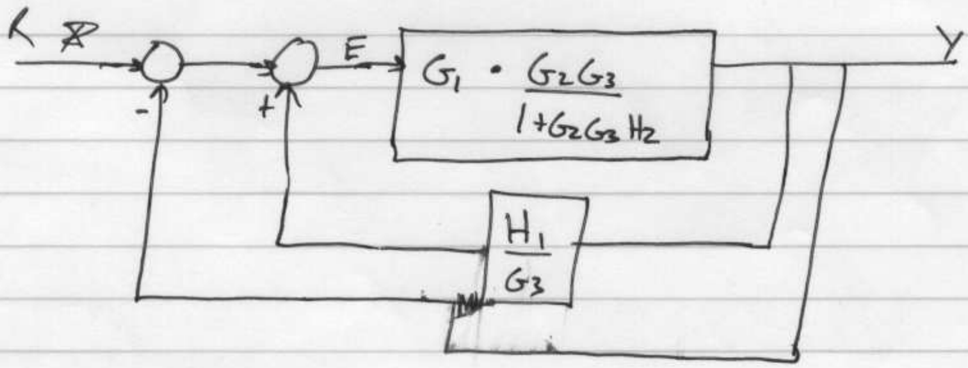
example



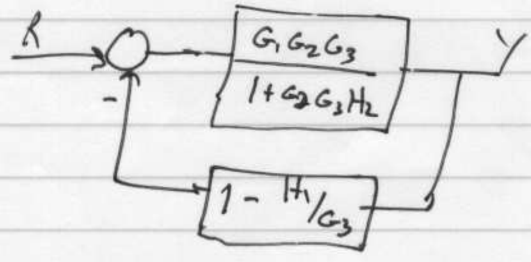




$$= \frac{G_2G_3}{1+G_2G_3H_2}$$



$$E = R_+ (H_1/G_3)Y - Y = R - (1 - H_1/G_3)$$



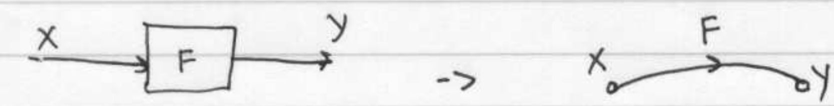
$$\frac{Y}{R} = \frac{\dfrac{G_1G_2G_3}{1+G_2G_3H_2}}{1 + \dfrac{G_1G_2G_3}{1+G_2G_3H_2} * \left(1 - \dfrac{H_1}{G_3}\right)}$$

$$= \frac{G_1G_2G_3}{1+G_2G_3H_2 - G_1G_2H_1 + G_1G_2G_3}$$

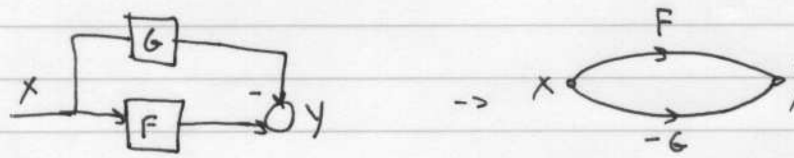
all factors in den are loops in the original block diagram

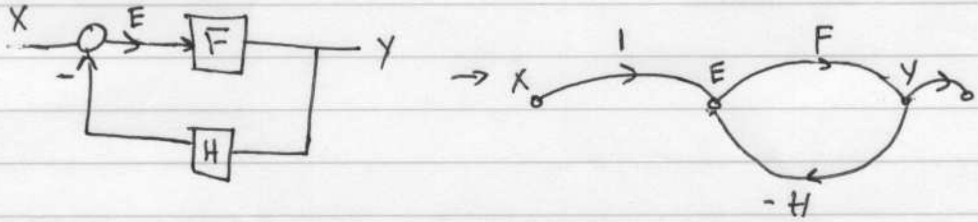
Signal Flow Graph

Mason's Formula



$Y = xf$

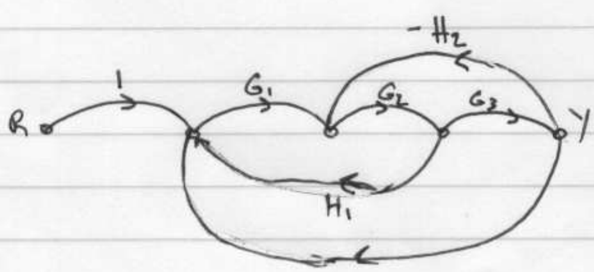




$T.F = \dfrac{F}{1+FH}$

	Block Diagrams	SFG
Signals	Lines	Nodes
Transfer Functions	Blocks	Branches

Last Day



Loops

$L_1 = G_1 G_2 H_1$

$L_2 = -G_1 G_2 G_3$

$L_3 = -G_2 G_3 H_2$

Forward Path

$P_1 = G_1 G_2 G_3$

$$\frac{Y}{R} = \frac{P_1}{1 - L_1 - L_2 - L_3}$$

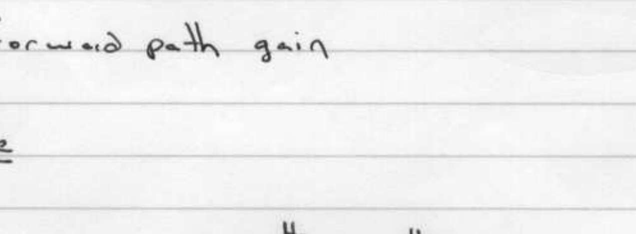
$$\frac{Y}{R} = \frac{\sum P_k \Delta_k}{\Delta}$$

$\Delta = 1 - \sum$ all loop gains $+ \sum$ all loop gain products of 2 non-touching loops $- \sum$ all loop gain products of 3 non touching loops + . .

Δ_k : determinant (Δ) of the graph when the k_{th} path is eliminated.
→ eliminate all nodes

P_k = forward path gain

<u>example</u>



$P_1 = G_1 G_2 G_3 G_4$ $\qquad L_1 = G_2 H_2$ $\qquad L_3 = G_6 H_6$

$P_2 = G_5 G_6 G_7 G_8$ $\qquad L_2 = G_3 H_3$ $\qquad L_4 = G_7 H_7$

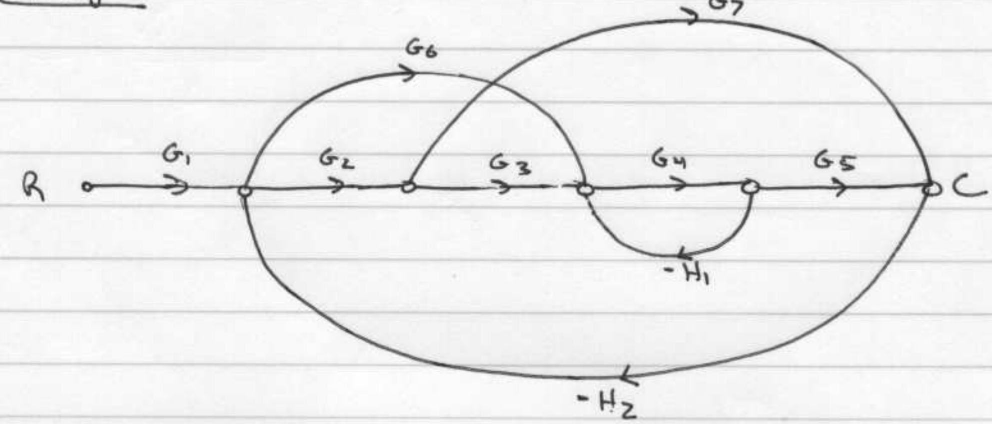
$$\Delta = 1 - L_1 - L_2 - L_3 - L_4 + L_1 L_3 + L_1 L_4 + L_2 L_3 + L_2 L_4$$

$\Delta_1 = 1 - L_3 - L_4$

$\Delta_2 = 1 - L_1 - L_2$

$$\frac{Y}{R} = \frac{P_1 \Delta_1 + P_2 \Delta_2}{\Delta}$$

example



$P_1 = G_1 G_2 G_3 G_4 G_5$ $\qquad$ $Loop_1 = -G_4 H_1$

$P_2 = G_1 G_6 G_4 G_5$ $\qquad$ $L_2 = -G_2 G_3 G_4 G_5 H_2$

$P_3 = G_1 G_2 G_7$ $\qquad$ $L_3 = -G_6 G_4 G_5 H_2$

$L_4 = -G_2 G_7 H_2$

$\Delta = 1 - L_1 - L_2 - L_3 - L_4 + L_1 L_4$

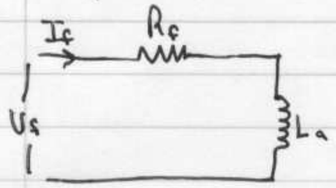
$\Delta_1 =$ nothing $= 1$

$\Delta_2 = 1$

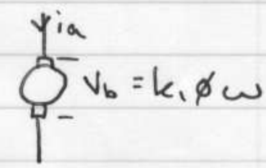
$\Delta_3 = 1 - L_1$

$$\frac{C}{R} = \frac{P_1 + P_2 + P_3 \Delta_3}{1 - L_1 - L_2 - L_3 - L_4 + L_1 L_4} \rightarrow \frac{P_1 + P_2 + P_3 \Delta_3}{\Delta}$$

DC Motor Modelling



$\phi = f(I_f)$
$= k_f I_f$



$V_b = k_1 k_f i_f \omega$

$P_e = V_b i_a \quad P_{mech} = T\omega \quad T = \text{torque}$

$T\omega \approx V_b i_a$

$T\omega = k_1 \phi \omega i_a$

$T = k_1 \phi i_a$

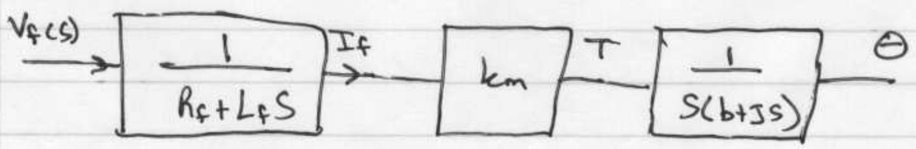
$$\boxed{T = k_1 k_f i_f i_a}$$

$I_a = \text{const}$ field controlled

$V_f = R_f i_f + L_a \frac{di_f}{dt} \to V_f(s) = R_f I_f + L_f S I_f = (R_f + L_f S) I_f$

$V_f \cdot \frac{1}{R_f + L_f S} = I_f \qquad T = (\underbrace{k_1 k_f i_a}_{k_m}) i_f$

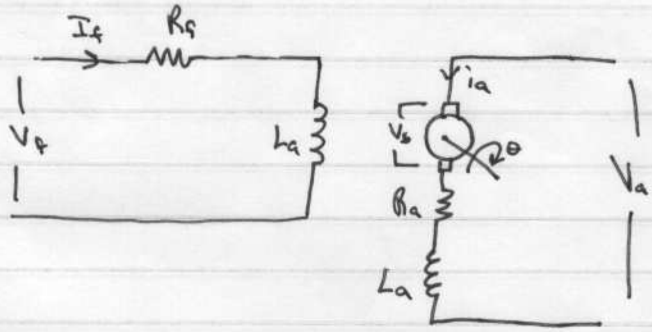
$J \to$ inertia
$b \to$ friction const.



$T - T_{fric} = J \frac{d^2\theta}{dt^2} \to T - \frac{b\,d\theta}{dt} = J \frac{d\theta^2}{dt^2}$

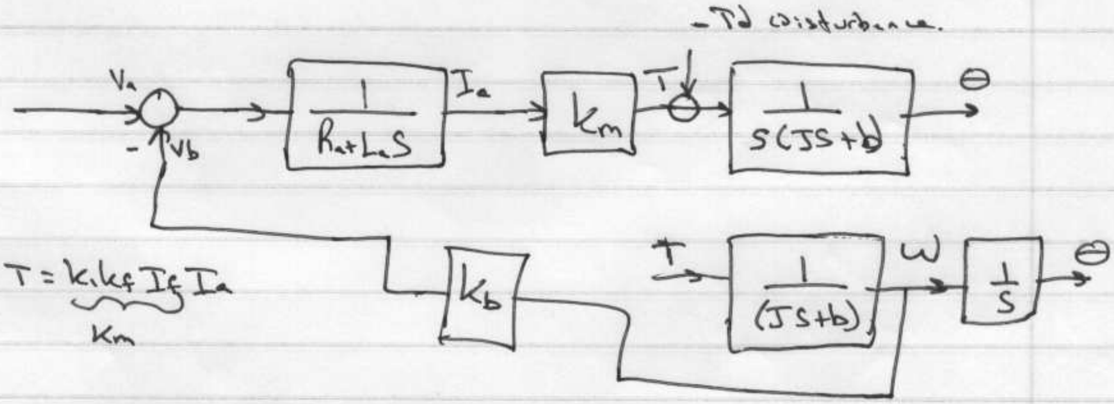
$T(s) = (b\theta S)s + J s^2 \theta s \qquad \frac{T(s)}{Js^2 + bs} = \theta(s)$

$i_f = \text{const}$



V_a - input
I_f - const

$$V_a - V_b = R_a i_a + L_a \frac{d i_a}{dt} \rightarrow V_a(s) - V_b(s) = R_a I_a + L_a S I_a$$



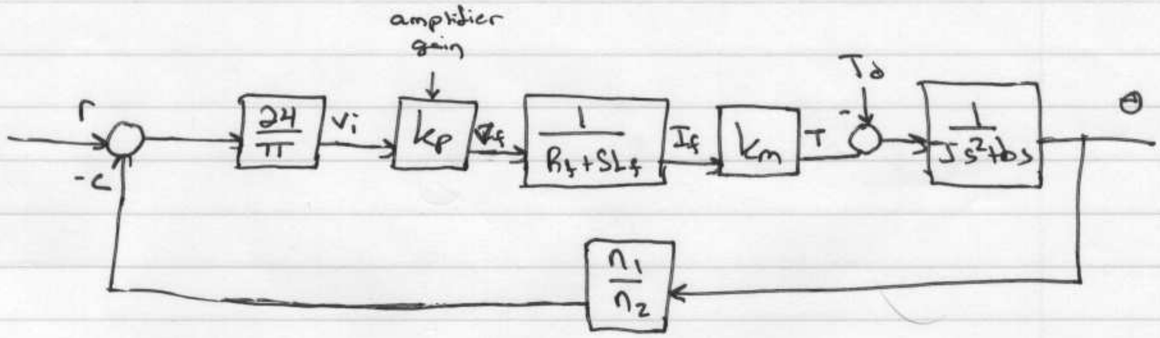
$$T = \underbrace{k_1 k_f I_f}_{K_m} I_a$$

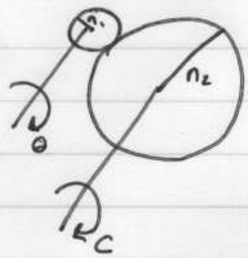
$$T = J\frac{d\omega}{dt} + f\omega$$

$$T(s) = JS\,\omega(s) + b\,\omega(s) \rightarrow \frac{T(s)}{Js+b} = \omega(s)$$

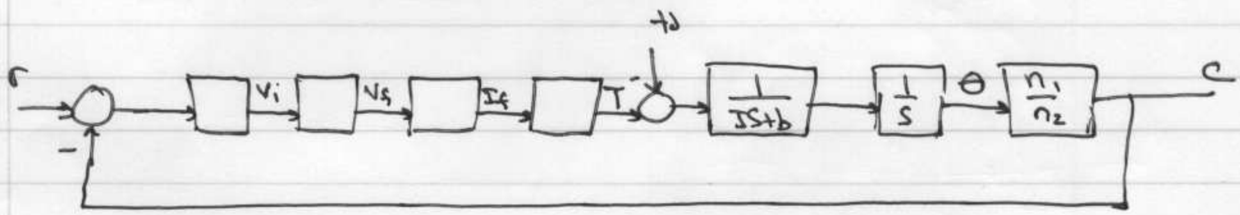
$$V_b = k_1 \phi \omega = \underbrace{(k_1 k_f i_f)}_{k_b}\omega$$

Example: Fig 4-49



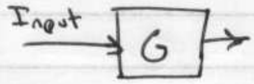


$$\frac{C}{\theta} = \frac{n_1}{n_2}$$



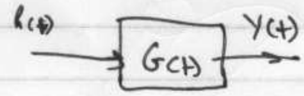
Time Response and Stability

Compare the performance of systems.



Types of inputs: step* - ramp
impulse - sinusoidal

Impulse Response



$Y(s) = R(s)\,G(s)$

Take $R(s) = 1$

$Y(s) = R(s)G(s) = G(s)$

$\underbrace{Y(t) = G(t)}_{\text{impulse response}}$

example

What is the impulse response?

$r(t) = \delta(t)$

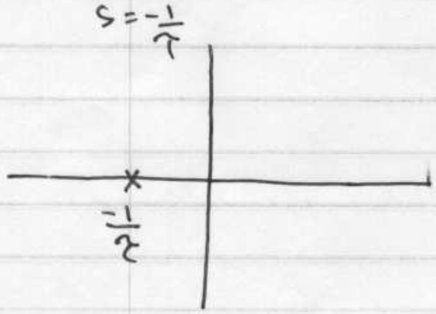
$R(s) = 1$

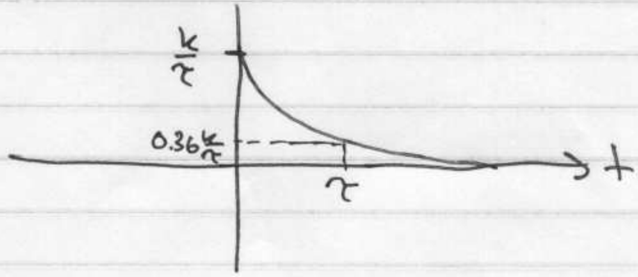
K - DC gain
τ - time constant

$$Y(s) = \frac{k}{\tau s+1} \cdot \frac{k/\tau}{s+1/\tau}$$

$$Y(t) = \frac{k}{\tau} e^{-\frac{t}{\tau}} \quad 0 \le t$$

$\tau s+1=0$
$s=-\frac{1}{\tau}$





if system is fast, τ is small, so pole is far from origin.
" slow, " " large, " " " close to origin

```
num = [K/τ]
den = [1  1/τ]
sys = tf(num,den)
impulse(sys,T)
```

Stable System
→ if the input is always bounded, the output is always bounded.

A signal $v(t)$, $t \geq 0$ is bounded if $\exists\, M > 0$ (there exists)
such that $|v(t)| \leq M,\ \forall t$ (for all t) (M: uniform bound)

A system G is is bounded input, bounded output stable (BIBO) if $y(t)$ is bounded whenever $r(t)$ is bounded.

$$r(t) \longrightarrow \boxed{G} \longrightarrow y(t)$$

Theorem: G is BIBO stable if the impulse response $g(t)$ is absolutely integrable, ie $\int_0^\infty |g(t)|\,dt < \infty$

Sufficiency $\int_0^\infty |g(t)|\,dt < M$, $y(t)$ will be bounded for all bounded inputs

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow Y(s) \qquad Y(s) = R(s)\,G(s)$$

$$y(t) = \int_{-\infty}^{t} g(t-\tau)\,r(\tau)\,d\tau$$

$$= \int_0^{t} g(t-\tau)\,r(\tau)\,d\tau$$

because $r(\tau) = 0 \quad \tau < 0$

$r(\tau) < M$

$|y(t)| \leq \int_0^t g(t-\tau)\, M \, d\tau$

$|y(t)| \leq M \int_0^t |g(t-\tau)| \, dt$

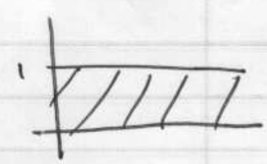
$|y(t)| \leq M \underbrace{\int_0^\infty |g(t-\tau)| \, d\tau}_{M_1}$ -> max value is $\int_0^\infty$

$|y(t)| \leq M M_1$

---

example $R(s) \rightarrow \boxed{\frac{1}{s}} \rightarrow Y(s)$ $\quad g(t) = u(t)$

$$\int_0^\infty |g(t)| \, d(t) = \int_0^\infty u(t) \, dt = \infty$$

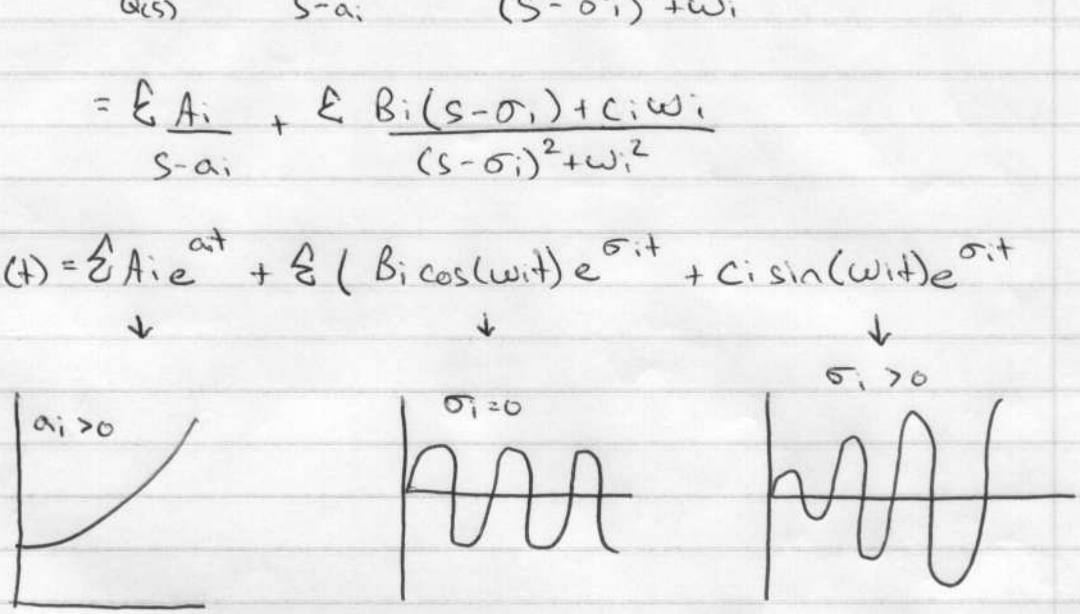


If you have an integrator in your system, and the input is constant, the system is not stable.

$$G(s) = \frac{P(s)}{Q(s)} = \sum \frac{A_i}{s-a_i} + \sum \frac{B_i s + C_i}{(s-\sigma_i)^2+\omega_i^2}$$

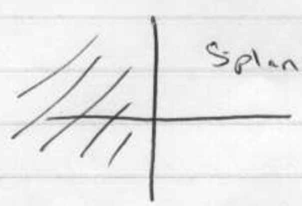
$$= \sum \frac{A_i}{s-a_i} + \sum \frac{B_i(s-\sigma_i)+C_i\omega_i}{(s-\sigma_i)^2+\omega_i^2}$$

$$g(t) = \sum A_i e^{a_i t} + \sum \left(B_i \cos(\omega_i t) e^{\sigma_i t} + C_i \sin(\omega_i t) e^{\sigma_i t} \right.$$

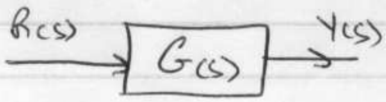


A system is BIBO stable only if all poles are stable.

$Re\ s < 0$



<u>Step Response</u>



$R(s) = \frac{1}{s}$ $\qquad Y(s) = \frac{1}{s} G(s)$

$y(t) \int_0^T g(t-\tau)\, d\tau$

$$Y(\infty) = \int_0^\infty g(t-\tau)\,d\tau$$
$$= \int_0^\infty g(t)\,dt$$

$$G(s) = \mathcal{L}\,g(t) = \int_0^\infty g(t)e^{-st}\,dt$$
$$G(0) = \int_0^\infty g(t)\,dt$$
$$Y(\infty) = G(0)$$

Assume $G(s)$ is stable. The DC gain of $G(s)$ is $G(0)$.

$$\frac{1}{s} \longrightarrow \boxed{\frac{k}{\tau s+1}} \longrightarrow Y(s)$$

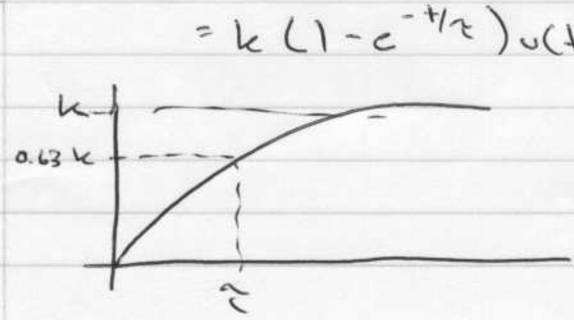
$\rightarrow$ steadystate output is K

$$Y(s) = \frac{1}{s}\cdot\frac{k}{\tau s+1} = \frac{A}{s} + \frac{B}{\tau s+1} = \frac{A}{s} + \frac{B}{s+1/\tau} = \frac{1}{s} + \frac{k/\tau}{s+1/\tau}$$

$$A = \frac{k/\tau}{1/\tau} = k \qquad B = \left.\frac{k/\tau}{s}\right|_{s=-1/\tau} = -k$$

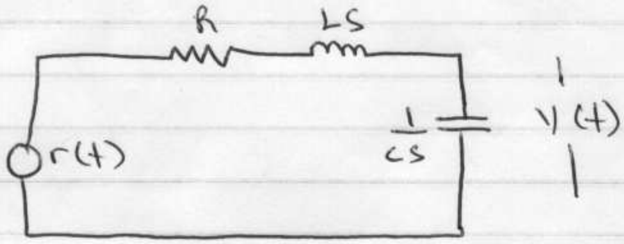
$$Y(s) = k\left(\frac{1}{s} - \frac{1}{s+1/\tau}\right)$$

$$y(t) = k(u(t)) - ke^{-t/\tau}u(t)$$
$$= k(1-e^{-t/\tau})u(t)$$

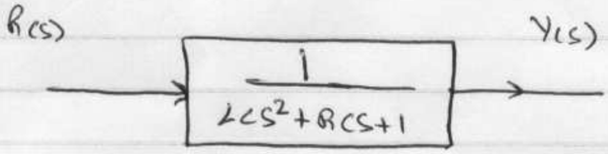


Second Order

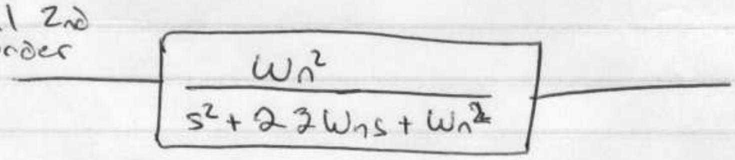
Example



$$\frac{Y(s)}{R(s)} = \frac{\frac{1}{CS}}{R + LS + 1/CS} = \frac{1}{LCS^2 + RCS + 1}$$



general 2nd Order



$$\frac{\frac{1}{LC}}{s^2 + \frac{R}{L}s + \frac{1}{LC}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$$\omega_n = 1/\sqrt{LC}$$

$$\frac{R}{L} = 2\zeta\omega_n = 2\zeta \, 1/\sqrt{LC}$$

$$\zeta = \frac{1}{2}R\sqrt{C/L}$$

Midterm
Friday, Oct 22. Physics 103 8:00 - 9:20

Second Order Systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

To ~~deam~~ determine if the system is stable, we must find the poles of the transfer function.

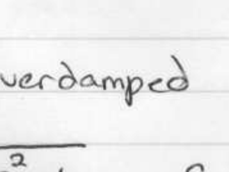
$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

$$s = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

Case 1

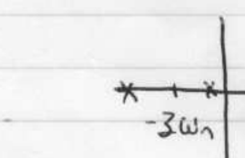
$\zeta > 1$ overdamped -> two real solutions

$$s = -\zeta\omega_n \pm \omega_n\underbrace{\sqrt{\zeta^2 - 1}}_{\approx \zeta}, \qquad s_1 < -\zeta\omega_n + \zeta\omega_n = 0$$
$$s_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$



Case 2

$\zeta = 1$ $s_{12} = -\zeta\omega_n$ Critically Damped

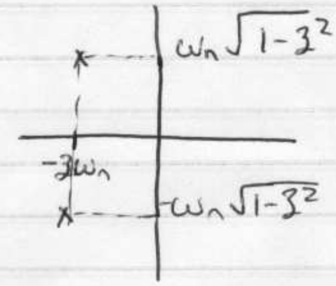


EE 481
Oct 1, 2004
2/3

Case III

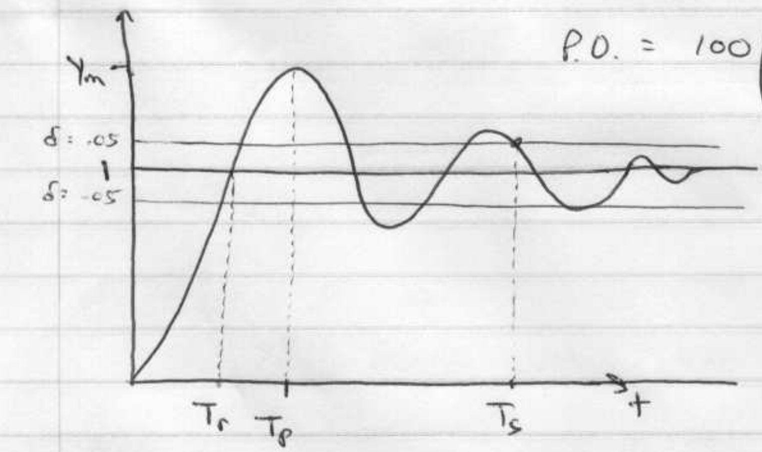
$\zeta < 1$ underdamped

$$S_{1,2} = -\zeta\omega_n \pm \omega_n j\sqrt{1-\zeta^2}$$



* System will always be stable when $\zeta > 0$

Step Response Specs



$$P.O. = 100\left(\frac{Y_m - Y(\infty)}{Y(\infty)}\right) = (Y_m - 1)\,100 \rightarrow \text{percent overshoot}$$

T_p = peak time

Rise time: T_R -> underdamped: Time for $Y(t)$ to go from 0 to $Y(\infty)$
overdamped: Time for $y(t)$ to go from 0.1 to $0.9\,y(\infty)$

Settling time: T_S (with tolerance δ) -> minimum time to satisfy
$|y(t) - y_\infty| \leq \delta Y_\infty \quad \forall t \geq T_S$

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

1) P.O. and T.P. exist for $0 \leq \zeta \leq 1$

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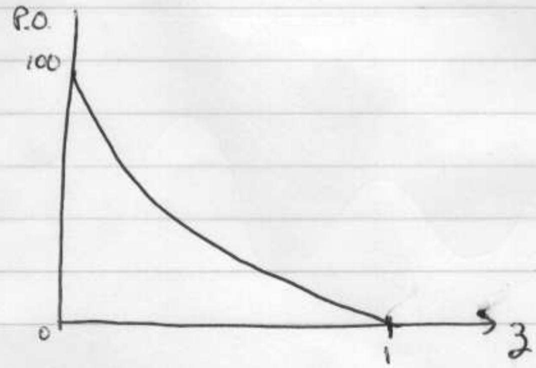
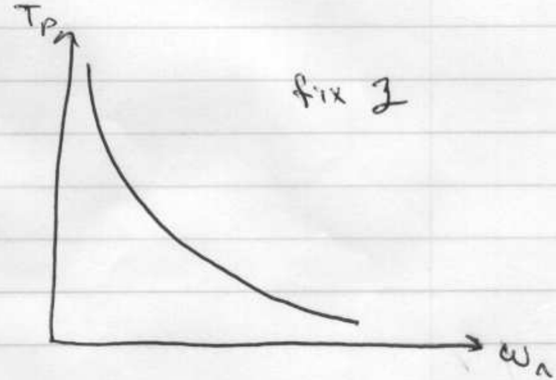
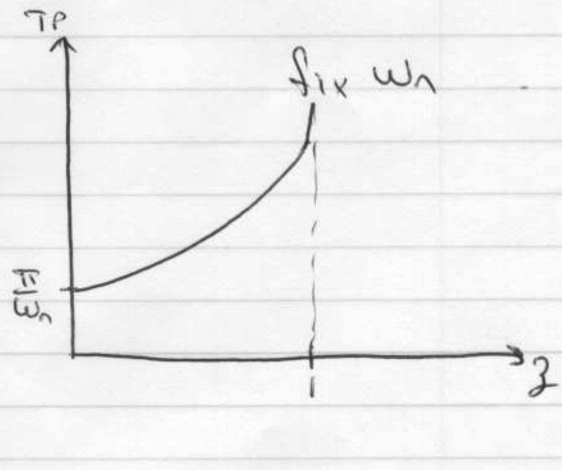
$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta \omega_n t} \sin(\beta \omega_n t + \theta)$$

$$P.O. = \exp\left\{ \frac{-\zeta \pi}{\sqrt{1-\zeta^2}} \right\} \times 100$$

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}}$$

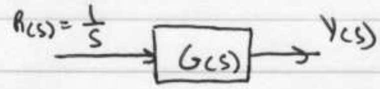




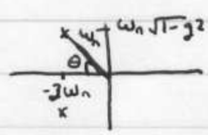


Step Response of Second Order Systems

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Under damped $0 < \zeta < 1$
-> two conjugate poles



$\theta = \cos^{-1}\zeta$
distance from origin = ω_n

Critically damped $\zeta = 1$

Over damped $\zeta > 1$
-> two real poles

From Last Day

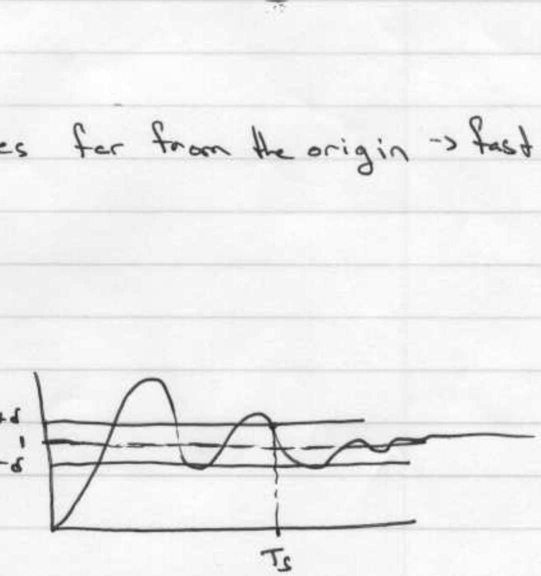
$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

-> Poles far from the origin -> fast system

$$P.O. = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}}$$

$$T_s \quad \left|\frac{1}{\beta} e^{-\zeta\omega_n t}\sin(\beta t + \theta)\right| < \delta$$

$$\beta = \sqrt{1-\zeta^2}$$



or $\frac{1}{\beta} e^{-\zeta\omega_n t} < \delta$, $T_s \leq T$

$$e^{-\zeta\omega_n t} < \beta\delta$$

$$-\zeta\omega_n t < \ln(\beta\delta)$$

$$T_s < -\frac{\ln(\delta) + \ln(\beta)}{\zeta\omega_n}$$

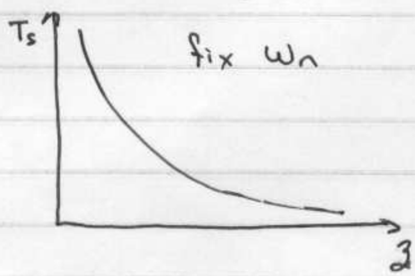
$$T_s < -\frac{\ln(\delta) + \frac{1}{2}\ln(1-\zeta^2)}{\zeta\omega_n}$$

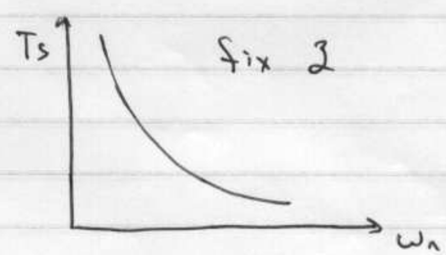
$\delta = 2\%$

$$T_s = \frac{3.912 - \frac{1}{2}\ln(1-\zeta^2)}{\zeta\omega_n}$$

$$T_s \approx \frac{4}{\zeta\omega_n} \quad : \delta = 2\%$$

for $\delta = 5\%$ $\quad T_s \approx \dfrac{3}{\zeta\omega_n}$





Remarks:

If we fix ω_n and increase ζ, our P.O. will decrease and T_s will decrease, However T_r and T_p will increase.

If we fix ζ, and increase ω_n, our P.O. will not change, T_s will decrease, T_r will decrease and T_p will decrease.

There is a trade off : a small P.O. gives a large ζ which gives a large T_p

We want $0.4 \leq \zeta \leq 0.8$ and $1.5\% \leq P.O. \leq 25\%$

Example

$P.O. \leq 25\%$; $T_s \leq 1$ sec , $\delta = 2\%$

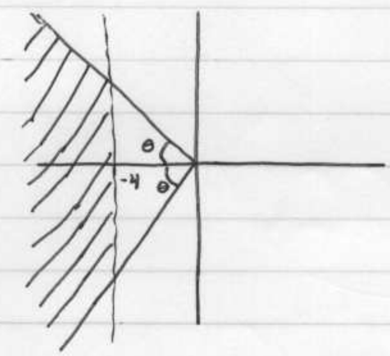
Find the location of the poles.

Using graph in text

$P.O. \leq 25\% \Rightarrow \boxed{\zeta \geq 0.4}$

$\dfrac{4}{\zeta\omega_n} \leq 1 \Rightarrow \boxed{4 \leq \zeta\omega_n}$

$\theta = \cos^{-1}\zeta$



Adding extra poles to a second order system.

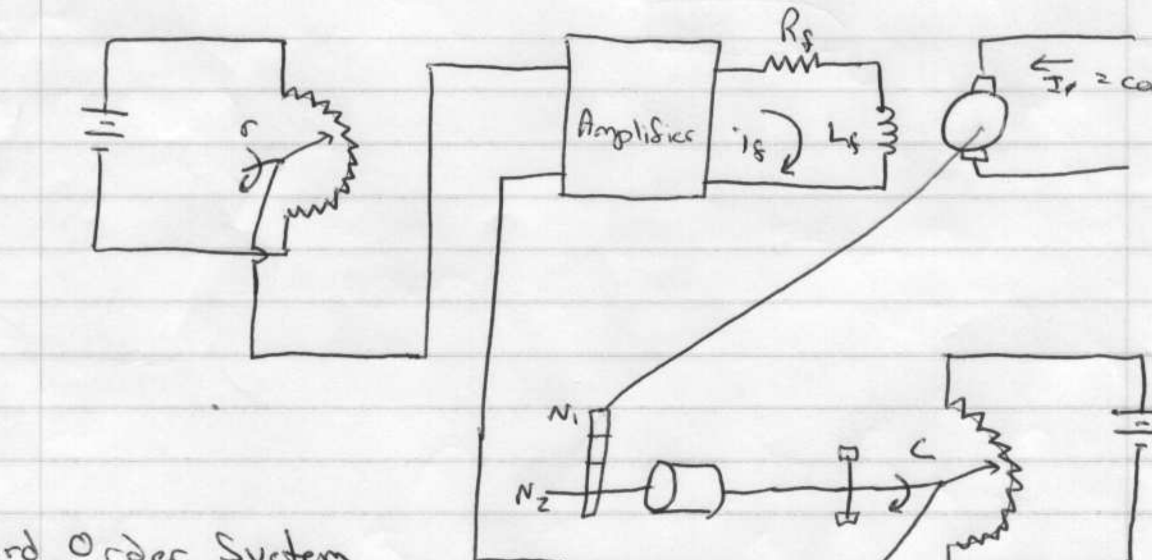
if $\left|\frac{1}{\gamma}\right| \geq 10|\zeta\omega_n|$ then

$$\frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)(\gamma s+1)} \approx \frac{\omega_n^2}{(s^2+2\zeta\omega_n s+\omega_n^2)}$$

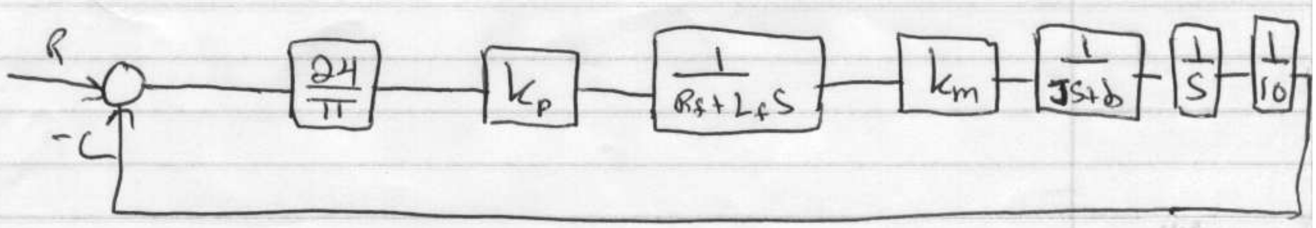
Adding extra zeros to a second order system

$$\frac{\alpha s+1}{s^2+2\zeta\omega_n s+\omega_n^2} \quad : \alpha \text{ can be +ve or -ve} \qquad s=\frac{-1}{\alpha}$$

There is an undershoot if α is -ve
If this zero is far from the origin compared to the poles, it can be ignored.



3rd Order System.



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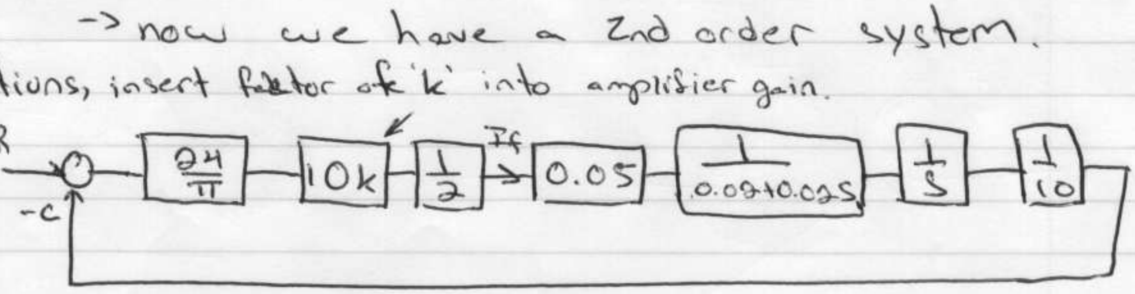
$k_i = \frac{24}{\pi}$ gain of potentiometer error detector.
$K_p = 10$ amplifier gain
$R_f = 2\Omega$ field winding resistance
$L_f = 0.1H$ field winding inductance
$K_m = 0.05$
$n = 1/10$ gear ratio
$J = 0.02\ kg\,m^2$ moment of inertia, reference to motor shaft
$b = 0.02$

Poles

$2 + 0.1s = 0 \rightarrow s = -20 \rightarrow$ very large
$0.02 + 0.02s = 0 \rightarrow s = -1$
$s = 0$

now $\frac{1}{R_f + L_f s} = \frac{1}{2 + \cancel{0.1s}} = \boxed{\frac{1}{2}}$

-> now we have a 2nd order system.
To reduce Oscillations, insert factor of 'k' into amplifier gain.



$$\frac{C}{R} = \frac{\frac{191}{20}k}{s^2 + s + \frac{191k}{20}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n + \omega_n^2}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.5} = 8s$$

Find k so that P.O. $= 5\%$ ($d = 2\%$)

$\zeta = 0.7$ (from graph) $\quad \left.\frac{4}{\zeta\omega_n} = 8\right|_{\zeta=0.7} \rightarrow \omega_n = 0.71 \quad \left.\frac{191k}{20} = \omega_n^2\right|_{\omega_n=0.71} \quad \boxed{k = 0.05}$

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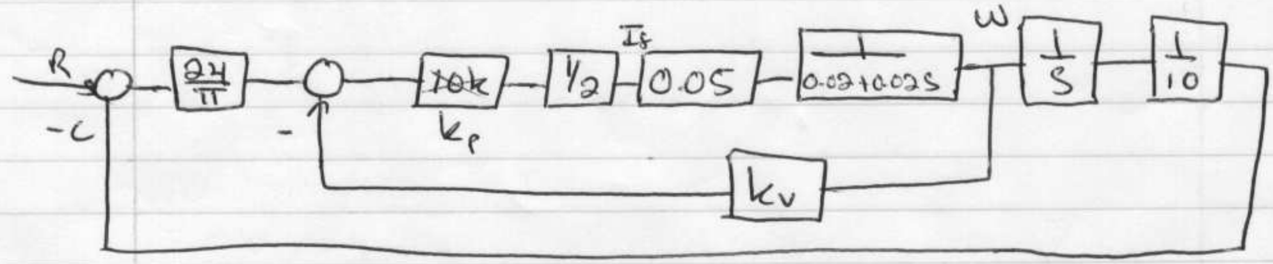


Now we want $PO = 5\%$
$T_s = 2s$

$$\frac{R}{C} = \frac{\frac{3}{\pi} k_p}{s^2 + s(1 + 1.25 k_p k_v) + \frac{3}{\pi} k_p} =$$

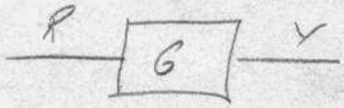
$$T_s = 2s \rightarrow T_s = \frac{4}{\zeta \omega_n} \rightarrow \zeta \omega_n = 2$$

$$P.O. = 5\% \rightarrow \zeta = 0.7 \qquad \omega_n = \frac{2}{0.7} \qquad \frac{3}{\pi} k_p = \omega_n^2 = \frac{2^2}{.7^2}$$

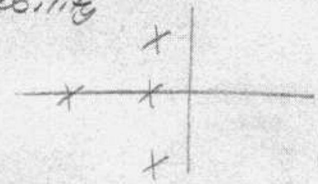
$$2\zeta\omega_n = 1 + 1.25 k_p k_v = 4 \qquad \boxed{k_p = 8.542}$$

$$k_v = \frac{4-1}{1.25 k_P} = \frac{3}{1.25 \cdot \frac{4\pi}{(.49)3}} = \frac{3}{(1.25)(8.542)} = \boxed{k_v = 0.281}$$

Closed Loop Stability



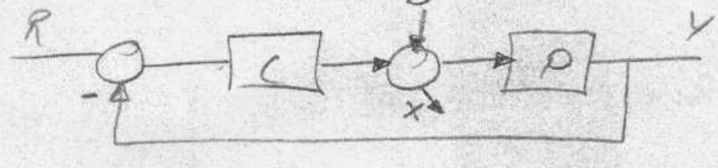
Stability



$g(t) \xrightarrow{\mathcal{L}} G(s)$

$\int_0^\infty |g(t)|\,dt \Longleftrightarrow$ BIBO stable

<u>Example</u> STILL STABLE if $D=1$?



$P = \frac{1}{s-1}$

$C = \frac{s-1}{s+5}$

$$\frac{Y}{R} = \frac{CP}{1+CP} = \frac{\left(\frac{s-1}{s+5}\right)\left(\frac{1}{s-1}\right)}{1+\left(\frac{1}{s+5}\right)} = \frac{\frac{1}{s+5}}{1+\frac{1}{s+5}} = \frac{1}{s+5+1} = \frac{1}{s+6}$$

$\therefore$ there is a pole @ $s = -6$

$$\frac{Y}{D} = \frac{P}{1+PC} = \frac{\frac{1}{s-1}}{1+\frac{1}{s+5}} = \frac{\frac{1}{s-1}}{\frac{s+6}{s+5}} = \frac{s+5}{(s-1)(s+6)}$$

$\therefore$ System is <u>not stable</u> because $(s-1)$ means that there is a pole in the RHP of s plane.

Definition

The feedback system is stable if the transfer function from the inputs (R, D) to (Y, X) are stable

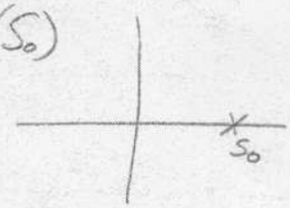
$$\frac{Y}{R}, \frac{Y}{D}, \frac{X}{R}, \frac{X}{D}$$

$$\frac{Y}{R} = \frac{CP}{1+CP} \; ; \; \frac{Y}{D} = \frac{P}{1+CP}$$

$$\frac{X}{R} = \frac{C}{1+CP} \; ; \; \frac{X}{D} = \frac{-PC}{1+PC}$$

① Assume the plant has an unstable pole (S_0)

$$P(s_0) = \infty$$



TAKE a $C(s)$ with a zero s_0, $C(s_0) = 0$

$$P(s_0)\, C(s_0) \neq \infty$$

<u>NOTE</u> $\frac{Y}{D}(s_0) = \frac{P(s_0)}{1 + P(s_0) C(s_0)} = \infty$

② TAKE $C(s_0) = \infty$ for an $S_0 > 0$

$P(s_0) = 0$

then $\frac{X}{R}(s_0) = \infty$

∴ ANY POLE and zero cancelation in $P(s)C(s)$ becomes a pole of either $\frac{P}{1+PC}$ OR $\frac{C}{1+PC}$

So closed loop stability is achieved only if:

① no unstable pole-zero cancellation

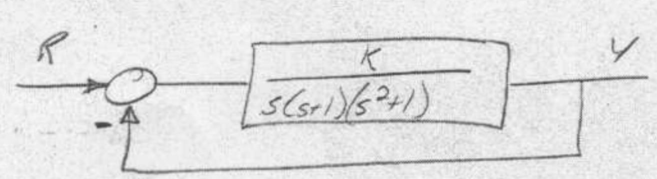
② roots of $1 + P(s)C(s) = 0$ are all stable

i.e. $Re\{s\} < 0$

EXAMPLE

$$P(s)C(s) = \frac{A(s)}{B(s)} \; ; \; 1 + P(s)C(s) = 0$$

$$1 + \frac{A(s)}{B(s)} = 0 \Rightarrow B(s) + A(s) = 0 \text{; Characteristic polynomial}$$



$$\frac{Y}{R} = \frac{P}{1+P} = \frac{\frac{K}{s(s+1)(s^2+1)}}{1 + \frac{K}{s(s+1)(s^2+1)}} = \frac{K}{s(s+1)(s^2+1) + K}$$

$s(s+1)(s^2+1) + K$

$s^4 + s^3 + s^2 + s + K \rightarrow$ ROOTS([1 1 1 1 K])

$$\left.\begin{array}{l} s = 0.309 \pm j0.9511 \\ \;\;\;\; -0.809 \pm j0.5878 \end{array}\right\} \text{if } K = 1$$

Routh-Hurwitz Stability Criterion

$\rightarrow \boxed{G} \rightarrow$ G is stable if $\int_0^\infty |g(t)|\,dt < \infty$ and if the roots of the denominator are in the left hand plane.

$$G(s) = \frac{Q(s)}{P(s)}$$

Take $P(s)$: some roots are real and some are complex

1) r_i
2) $\sigma_e \pm j\omega_e$

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + \dots + a_0$$

$$P(s) = a_n \left(\prod_i (s - r_i)\right) \prod_e (s - \sigma_e + j\omega_e)(s - \sigma_e - j\omega_e)$$

$$= a_n \prod_i (s - r_i) \prod_e ((s - \sigma_e)^2 + \omega_e^2)$$

$P(s)$ is stable when: $r_i, \sigma_e < 0$

$\therefore$ $P(s)$ is a polynomial with positive coefficients

ex)

$$G(s) = \frac{s^2 + 3s + 2}{s^3 + 4s^2 - 2s + 1}$$

$G(s)$ is not stable because the coefficients of the denominator $(s^3 + 4s^2 - 2s - 1)$ do not have the same sign.

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ex) $G(s) = \dfrac{s^2+3s+1}{s^3+0.25s^2+0.2s+1}$

Poles are $-1,\ 0.4 \pm j0.9165$

Not stable because $\sigma > 0$.

$G(s) = \dfrac{Q(s)}{P(s)}$

$P(s) = a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0$

$$\begin{array}{c|ccc}
s^n & a_n & a_{n-2} & a_{n-4} \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} \\
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} \\
\vdots & c_{n-1} & c_{n-3} & \\
s^0 & & &
\end{array}$$

$$b_{n-1} = \frac{\begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix}}{-a_{n-1}} \qquad b_{n-3} = \frac{\begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}}{-a_{n-1}}$$

$$c_{n-1} = \frac{\begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}}{-b_{n-1}} \qquad c_{n-3} \ \frac{\begin{vmatrix} a_{n-1} & a_{n-5} \\ b_{n-1} & b_{n-5} \end{vmatrix}}{-b_{n-1}}$$

$P(s)$ is stable if and only if there is no change of sign in the first column of the table

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ex)

$P_s = s^2 + a_1 s + a_0$

$$\begin{array}{c|cc} s^2 & 1 & a_0 \\ s & a_1 & 0 \\ 1 & b = a_0 & \end{array} \qquad b = \frac{(1)(0) - (a_1)(a_0)}{-a_1} = a_0$$

$P(s)$ is stable if a_0 and a_1 are > 0

ex)

$P(s) = s^3 + a_2 s^2 + a_1 s + a_0$

$$\begin{array}{c|cc} s^3 & 1 & a_1 \\ s^2 & a_2 & a_0 \\ s & b & 0 \\ 1 & c = a_0 & 0 \end{array} \qquad b = \frac{a_0 - a_1 a_2}{-a_2} \qquad c = \frac{-a_0 b}{-b} = a_0$$

$b > 0 \rightarrow \frac{a_1 a_2 - a_0}{a_2} > 0 \xrightarrow{as\ a_2 > 0} a_1 a_2 - a_0 > 0 \rightarrow a_1 a_2 > a_0$

$a_2 > 0$

$a_0 > 0$

$$\boxed{\begin{array}{l} a_1 a_2 > a_0 \\ a_2 > 0 \\ a_0 > 0 \end{array}}$$

ex)

$s^3 + 0.2s^2 + 0.2s + 1$

$a_2 = 0.2 > 0$ ✓

$a_0 = 1 > 0$ ✓

$a_1 a_2 = (0.2)(0.2) = 0.04 < 1$ ✗

not stable.

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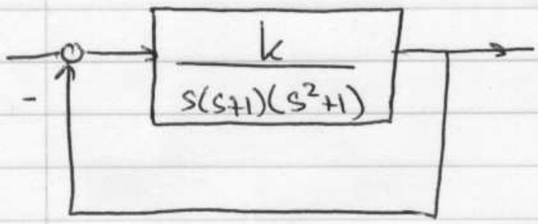
ex)



$$G(s) = \frac{\frac{k}{s(s+1)(s^2+1)}}{1 + \frac{k}{s(s+1)(s^2+1)}} = \frac{k}{s(s+1)(s^2+1)+k}$$

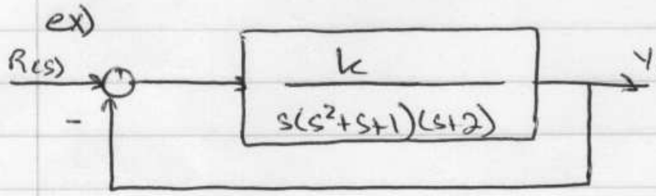
$$= \frac{k}{s^4 + s^3 + s^2 + s + k}$$

$$\begin{array}{c|ccc} s^4 & 1 & 1 & k \\ s^3 & 1 & 1 & \\ s^2 & b=0 & & \\ s & & & \\ 1 & & & \end{array}$$

$$b = \frac{(1)(1) - (1)(1)}{-1} = 0$$

Not stable!

ex)



Find the range of k for which the system is stable.

$$\frac{Y(s)}{R(s)} = \frac{\frac{k}{s(s^2+s+1)(s+2)}}{1 + \frac{k}{s(s^2+s+1)(s+2)}} = \frac{k}{s^4 + 3s^3 + 3s^2 + 2s + k}$$

$$\begin{array}{c|ccc} s^4 & 1 & 3 & k \\ s^3 & 3 & 2 & 0 \\ s^2 & +7/3 & k & 0 \\ s^1 & c_1 & 0 & \\ 1 & k & & \end{array}$$

$$c_1 = \frac{3k - \frac{14}{3}}{+7/3} > 0 \rightarrow 3k > \frac{14}{3}$$

$$k < \frac{14}{9}$$

$$\boxed{\begin{array}{l} k > 0 \\ k < 14/9 \end{array}} \rightarrow \boxed{\frac{14}{9} > k > 0}$$

Hilroy

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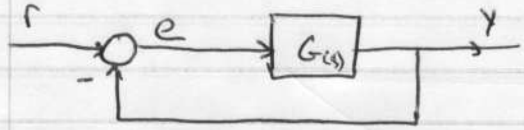
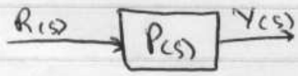
Y_H

Steady State Errors



$G(s) = \dfrac{A(s)}{s^n B(s)}$ n: type of system

ex) $G(s) = \dfrac{s+1}{s^2+s} = \dfrac{s+1}{s(s+1)} = \dfrac{1}{s}$: type of system is '1'



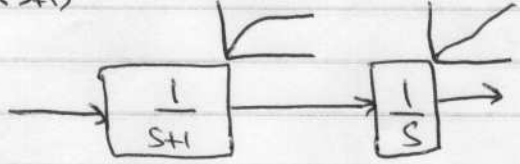
$Y(s) = P(s)R(s)$

if $P(s)$ is stable: $\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) = \lim_{s\to 0} sP(s)R(s)$

case 1 : Take $R(s) = \frac{1}{s}$ => step input

$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sP(s)\frac{1}{s} = \lim_{s\to 0} P(s) = P(0)$

$P(s) = \dfrac{1}{s(s+1)} \to \infty$



Closed Loop System -> assume it is stable

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)}$$

case: $R(s) = \frac{1}{s}$ -> step input

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{s\cdot 1/s}{1+G(s)} = \lim_{s\to 0} \frac{1}{1+G(s)} = \frac{1}{1+\lim_{s\to 0} G(s)} = \frac{1}{1+K_p}$$

If $G(s)$ is a type '1' system, the limit will be ∞

$$\text{Define } K_p = \lim_{s\to 0} G(s) = \begin{cases} \text{finite} & \text{type } 0 \\ \text{infinite} & \text{type} \geq 1 \end{cases}$$

for type 0 : $\lim_{t\to\infty} e(t) = \frac{1}{1+K_p}$

for type ≥ 1 : $\lim_{t\to\infty} e(t) = \frac{1}{1+\infty} = 0$

An integrator forces the output to converge to the input.
$\hookrightarrow \frac{1}{s}$

$$\text{Define } K_v = \lim_{s\to 0} sG(s) = \begin{cases} 0 & \text{type } 0 \\ \text{finite} & \text{type } 1 \\ \infty & \text{type} \geq 2 \end{cases}$$

case 2: $R(s) = \frac{1}{s^2}$ -> ramp input

$$\lim_{t\to\infty} e(t) = \lim_{s\to 0} \frac{sR(s)}{1+G(s)} = \lim_{s\to 0} \frac{s\cdot 1/s^2}{1+G(s)} = \lim_{s\to 0} \frac{1}{s(1+G(s))}$$

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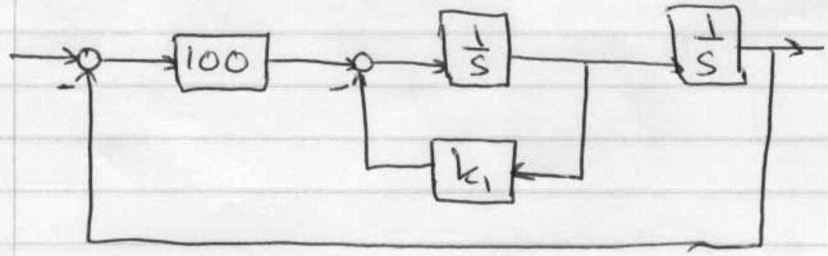
$$= \frac{1}{\lim_{s\to 0} s + sG(s)} = \frac{1}{0 + \lim_{s\to 0} sG(s)} = \boxed{\frac{1}{k_v} = e(\infty)}$$ ramp input

Integrator will allow output to follow output.
More than one integrator allows output to equal input.

Summary

Type	Unit Step	Ramp
0	$\frac{1}{1+k_p}$	∞
1	0	$\frac{1}{k_v}$
≥ 2	0	0

example



Is it possible to achieve: 1) P.O. in $y < 16\%$ if $r = u(t)$

2) T_s for $y < 1$ sec if $r = u(t)$, $\delta = 2\%$

3) $e_{ss} < 0.12$ if $r = tu(t)$

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1) $\exp\left(\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}\right) \leq 0.16 \rightarrow \zeta \geq 0.5$

$\zeta\omega_n = \frac{k_1}{2} \rightarrow \zeta = \frac{k_1}{20} \rightarrow \boxed{k \geq 10}$

forward Path T.F. $\rightarrow$ T.F of system $= \frac{100}{s^2+sk_1+100}$

2) $\frac{4}{\zeta\omega_n} \leq 1 \Rightarrow 4 \leq \zeta\omega_n$

$4 \leq \frac{k_1}{2} \rightarrow \boxed{8 \leq k_1}$

3) $k_v = \lim_{s\to 0} \frac{\cancel{s}\ 100}{(s+k_1)\cancel{s}} = \frac{100}{k_1}$

$\frac{1}{k_v} \leq 0.12 \rightarrow \frac{1}{100/k_1} \leq 0.12 \rightarrow \boxed{k_1 \leq 12}$

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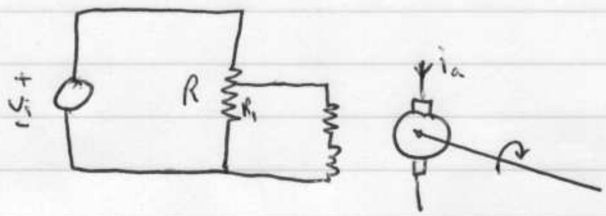
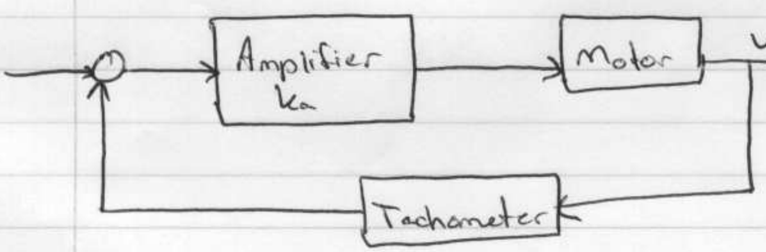
Effects of Feedback



motor

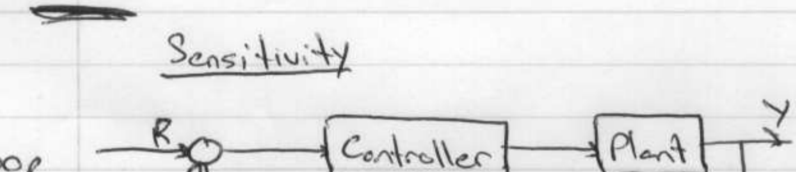
Theoretically you can control the speed of your motor with voltage. if you know everything about the motor.



Practically this works. You can have changes in your system and still have an accurate output.

Sensitivity

Closed Loop



$$T.F = \frac{Y}{R} = \frac{PC}{1+PC} \quad ; \quad S_P^{TF} = \frac{\left(\frac{\Delta T}{T}\right)}{\frac{\Delta P}{P}} = \frac{P}{T} \cdot \frac{\Delta T}{\Delta P} = \frac{P}{T}\frac{\partial T}{\partial P}$$

$$\frac{\partial T}{\partial P} = \frac{C(1+PC) - CPC}{(1+PC)^2} = \frac{C}{(1+PC)^2}$$

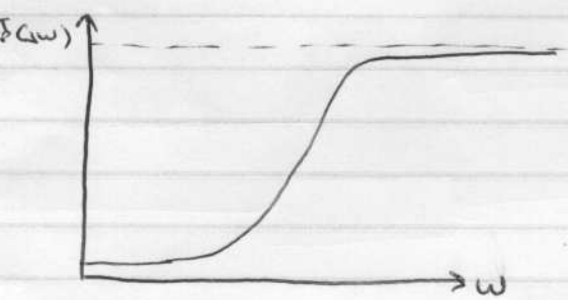
$$S_P^T = \frac{P}{\frac{PC}{1+PC}} * \frac{C}{(1+PC)^2} = \boxed{\frac{1}{1+PC} = S_P^T}$$

$$PC = \frac{N(s)}{D(s)}$$

Take the case where the degree of $N(s)$ is lower.

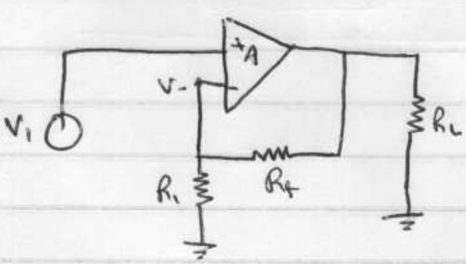
$$P(s)C(s)\Big|_{s=j\infty} = 0$$

$$S_P^T(j\infty) = \frac{1}{1+0} = 1$$

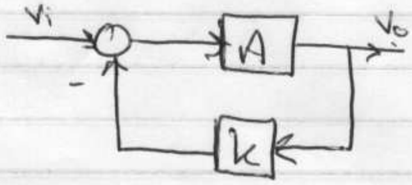


at a low frequency you have small sensitivity to changes in the plant.

example



$$V_- = \frac{V_o R_1}{R_1 + R_f} = V_o \cdot k$$



$$S_A^T = \frac{A}{T}\frac{\partial T}{\partial A} \qquad T = \frac{A}{1+Ak}$$

$$\frac{\partial T}{\partial A} = \frac{1+Ak-kA}{(1+Ak)^2} = \frac{1}{(1+Ak)^2}$$

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$$S^T_A = \frac{\partial T}{\partial A}\frac{A}{T} = \frac{1}{(1+Ak)^2}\cdot\frac{A}{A/(1+Ak)} = \boxed{\frac{1}{1+Ak} = S^T_A}$$

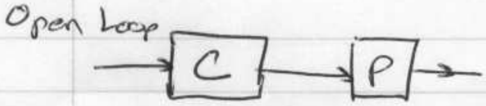
$A = 10^4$ $\quad S^T_A = \frac{1}{1+10^4 10^{-1}} = \frac{1}{1001} \approx 10^{-3}$ Negligable
$k = 0.1$

$$S^T_k = \frac{\partial T}{\partial k}\frac{k}{T} \; ; \; \frac{\partial T}{\partial k} = \frac{0 - A^2}{(1+Ak)^2} = \frac{-A^2}{(1+Ak)^2}$$

$$S^T_k = \frac{-A^2}{(1+Ak)^2} * \frac{k}{A/1+Ak} = \frac{-Ak}{(1+Ak)} \Rightarrow \frac{-10^4 10^{-1}}{(1+10^4 10^{-1})} \approx -1$$

$$\boxed{S^T_k = \frac{-Ak}{(1+Ak)} \approx -1}$$ very sensitive.

Open Loop

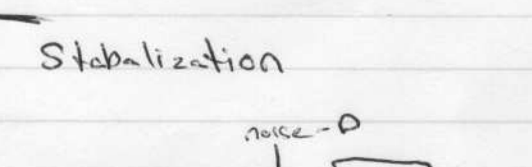


$T = PC$

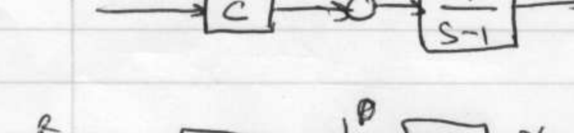
$$S^T_P = \frac{\partial T}{\partial P}\frac{P}{T} = \cancel{C}\cdot\frac{P}{\cancel{P}C} = 1$$ sensitive to changes in plant

---

Stabalization



not stable



$$\frac{Y}{R} = \frac{\frac{k}{s-1}}{1+\frac{k}{s-1}} = \frac{k}{s+(k-1)}$$

Pole $= -(k-1)$ stable when $(k-1) > 0$

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$$\frac{Y}{D} = \frac{\frac{1}{s-1}}{1+\frac{k}{s-1}} = \frac{1}{s+(k-1)}$$

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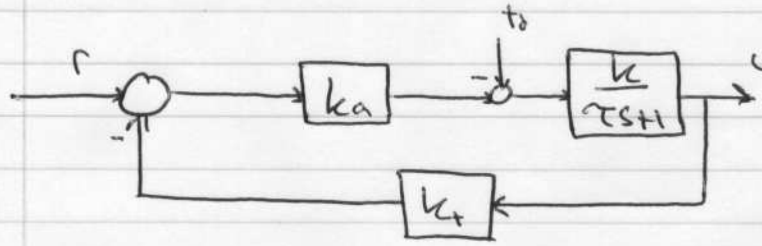
Properties of Feedback



$$\frac{r}{\omega} = \frac{\frac{k_a k}{\tau s+1}}{1+\frac{k_a k k_t}{\tau s+1}} = \frac{k_a k}{\tau s+1+k_a k k_t} = \frac{\frac{k_a k}{1+k_a k k_t}}{\left(\frac{\tau}{1+k k_a k_t}\right)s+1}$$

$$G_{(s)} = \frac{\omega}{t_d} = \frac{\frac{k}{\tau s+1}}{1+\frac{k k_a k_t}{\tau s+1}} = \frac{k}{\tau s+1+k k_a k_t}$$

$$G_{t_d}(0) = \frac{k}{1+k k_a k_t} \qquad \omega(\infty) = t_d(\infty)\frac{k}{1+k k_a k_t}$$

$\Rightarrow$ Feedback can reduce the effect of disturbance take $k k_a k_t >> 1$

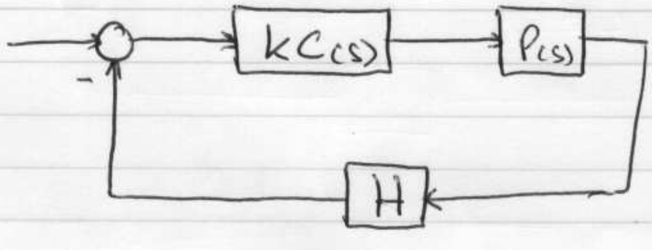
Summary

Openloop : Simple

Feedback (CL): Complex (controller, feedback)
Advantages : reduce sensitivity w.r.t. the plant
improve transient response
improve disturbance rejection
stabalize unstable plants

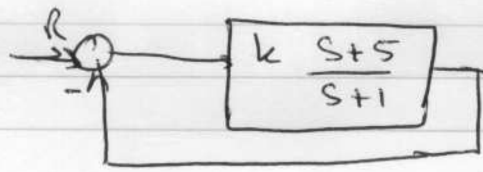
Root Locus

Root Locus



We want to find k so it is the best for the system.

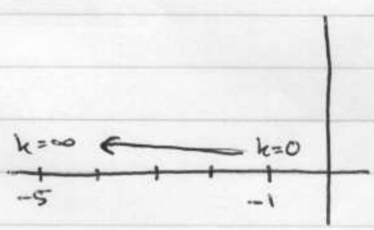
ex)



$$\frac{Y}{R} = \frac{k\frac{s+5}{s+1}}{1+k\frac{s+5}{s+1}} = \frac{k(s+5)}{(k+1)s+5k+1}$$

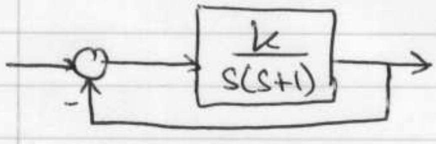
zero : -5

pole : $-\frac{5k+1}{1+k} = -\frac{5k+5-4}{1+k} = -5+\frac{4}{1+k} = S$

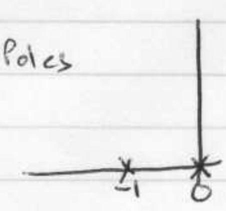


The curve startes at the pole of the transfer function and ends at the zero.

General Example



Poles



$$T(s) = \frac{\frac{k}{s(s+1)}}{1+\frac{k}{s(s+1)}} = \frac{k}{s^2+s+k}$$

$T(s) = \dfrac{k}{s^2+s+k}$ Find the poles of the C.L system as a function of k and plot on the s-plane.

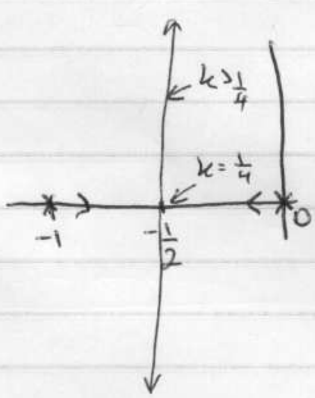
$s^2+s+k = 0$

$$s = \frac{-1 \pm \sqrt{1-4k}}{2}$$

when $1-4k = 0$, $k = 1/4$, $S_{1,2} = -1/2$

$k < 1/4$ -> two real poles $= \frac{-1}{2} \pm \frac{1}{2}\sqrt{1-4k}$

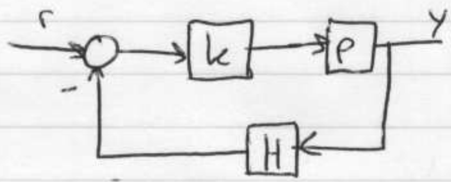
$k=0$; $s = -1, 0$



$k > \frac{1}{4}$ -> $S = -\frac{1}{2} \pm j\frac{1}{2}\sqrt{4k-1}$

Start at the poles and move to infinite (since there are no zeros)

Root Locus



$T(s) = \frac{Y(s)}{R(s)} = \frac{kP}{1+kPH}$

Find the roots as a function of k.

Poles: $1 + kPH = 0$

Define: $G = PH \;\rightarrow\; 1 + kG(s) = 0$

Assume: $G(s) = \frac{N(s)}{D(s)} \;\rightarrow\; 1 + k\frac{N(s)}{D(s)} = 0$

$D(s) + kN(s) = 0$ Characteristic Equation

when $k=0$: $D(s) = 0$
-> poles of open loop system are same as poles of closed loop system.

when $k=\infty$: $N(s) = 0$

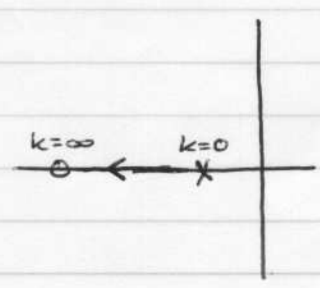
$\frac{D(s)}{k} + N(s) = 0$

-> the bounded poles of the closed loop system are equal to the zeros of the open loop system.

Take degree $(N) = n$
degree $(D) = d$ $\quad d \geq n$

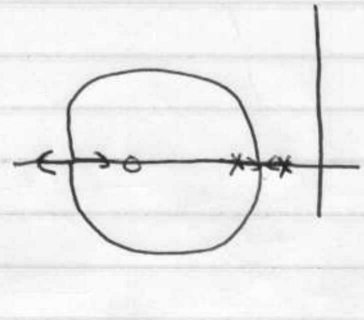
example

$G(s) = \frac{s+5}{s+2}$



example

$$G(s) = \frac{s+5}{(s+2)(s+3)}$$



So again, $1 + kG(s) = 0$

Question: Assume S_o is given, when is S_o on the root locus?

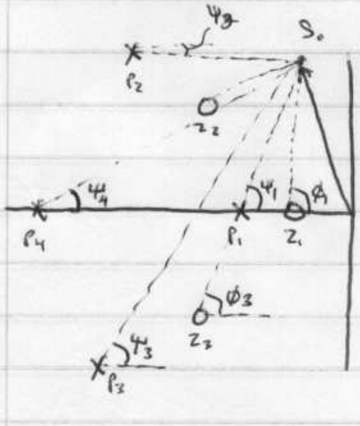
$1 + kG(s_o) = 0$? for $k \geq 0$

$G(s_o) = \frac{-1}{k}$ -> $G(s)$ must be a negative number.
so $G(s) \angle \pi$

$$G(s) = \frac{(s-z_1)(s-z_2)\ldots(s-z_n)}{(s-p_1)(s-p_2)\ldots(s-p_d)}$$

$$G(s_o) = \frac{(s_o-z_1)\ldots(s_o-z_n)}{(s_o-p_1)\ldots(s_o-p_d)}$$

$$\angle G(s_o) = \pi = [\angle(s_o-z_1) + \ldots + \angle(s_o-z_n)] - [\angle(s_o-p_1) + \ldots + \angle(s_o-p_d)]$$



$$\phi_1 + \phi_2 + \phi_3 - (\psi_1 + \psi_2 + \psi_3 + \psi_4) = \pi$$

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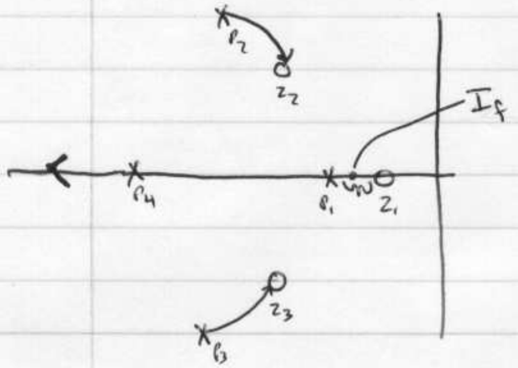


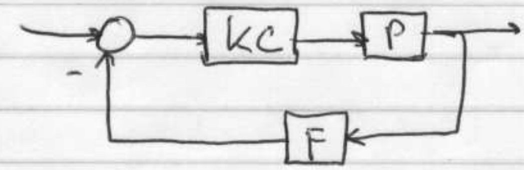
Is I_f on the root Locus?

$\angle Z_1$ to I_f = $180°$
$\angle Z_2$ to I_f = $-\angle Z_3$ to I_f -> cancel
$\angle P_1$ to I_f = 0
$\angle P_4$ to I_f = 0
$\angle P_2$ to I_f = $-\angle P_3$ to I_f -> cancel.

-> sum all the angles : 180 + 0 = 180 = π
-> yes it is on the root locus.

General Rule : if there is an odd number of poles and zeros on the right, then the point is on the root locus.
* all ~~conjugate~~ complex poles/zeroes must have a conjugate.

Root Locus

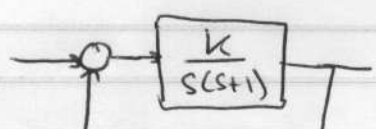


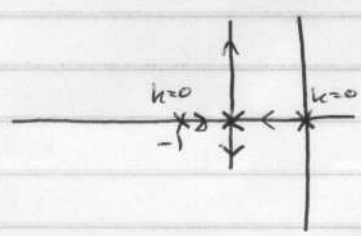
$CPF = G \;;\quad G(s) = \dfrac{N(s)}{D(s)}$

$1 + kCPF = 0 \Rightarrow 1 + kG(s) = 0 \rightarrow 1 + k\dfrac{N(s)}{D(s)} = 0$

$D(s) + kN(s) = 0$; degree of n and d

example





Must go towards ∞ because there are no zeroes.

$N(s) = (s-z_1)\ldots(s-z_n)$
$D(s) = (s-p_1)\ldots(s-p_d)$

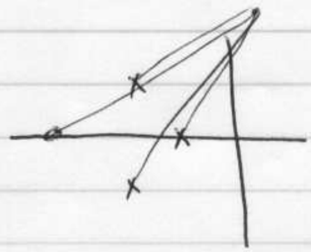
$1 + k\dfrac{N(s)}{D(s)} = 0 \rightarrow \dfrac{N(s)}{D(s)} = \dfrac{-1}{k}$

$\angle\left[\dfrac{(s-z_1)\ldots(s-z_n)}{(s-p_1)\ldots(s-p_d)}\right] = \pi$

$\angle s - z_i = \phi_i \qquad \angle s - p_i = \psi_i$

$\sum_{i=1}^{n} \phi_i - \sum_{i=1}^{d} \psi_i = \pi$

When the test point is far away, all the angles are close to each other.



$$\psi_i = \phi_i = \phi$$

$$n\phi - d\phi = \pi$$

$$\phi = \frac{\pi}{n-d} \rightarrow \frac{\pi}{1-3} = \frac{-\pi}{2}$$

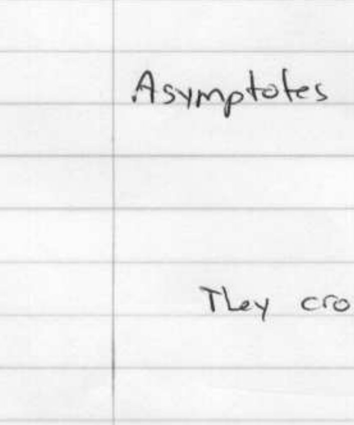
Asymptotes : The angles can be obtained by this relationship:

$$\angle s = \frac{180^\circ + k360^\circ}{n-m} = \frac{180^\circ + k\,360^\circ}{n-d}$$

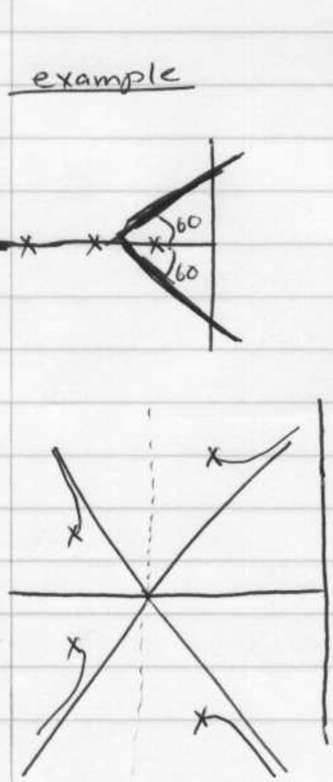
They cross the real line at $\sigma_c = \dfrac{(P_1 + \ldots + P_d) - (Z_1 + \ldots + Z_n)}{d-n}$

for $\dfrac{1}{s(s+1)} \rightarrow \sigma_c = \dfrac{0+(-1)}{2-0} = \dfrac{1}{2} \qquad \angle s = \dfrac{180}{0-2} = 90^\circ = \dfrac{\pi}{2}$

example



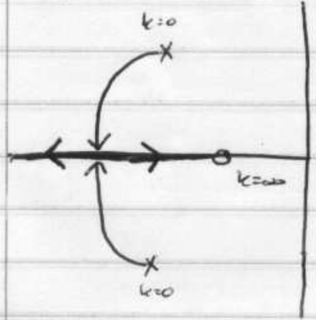
$$\angle s = \frac{180 + k360}{3-0} \Rightarrow \overset{k=0}{60}, \overset{k=1}{180}, \overset{k=2}{-60}$$

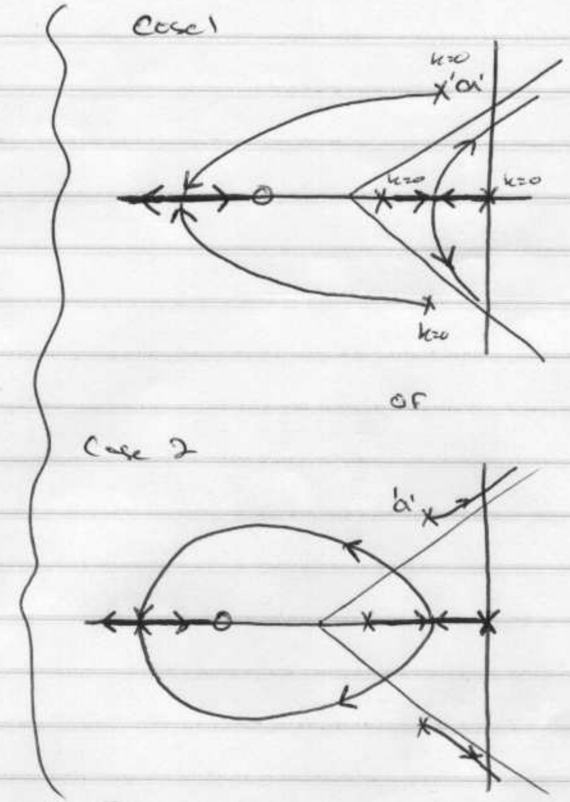


$$\angle s = \frac{180 + k360}{4-0} \Rightarrow \overset{k=0}{45}, 135, -135, \overset{k=3}{-45}$$

As k increases, it will become ustable.

ex)

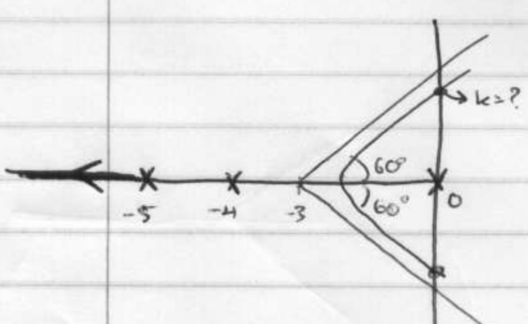




which one?
→ check the angle of departure of 'a', if it is 0°, then use case 2, if it is 180° then use case 1

example.

$$G(s) = \frac{1}{s(s+4)(s+5)}$$



$$\angle s = \frac{180 + k360}{3} = 60, 180, -60$$

$$\sigma_c = \frac{0 + (-4) + (-5)}{3-0} = \frac{-9}{3} = -3$$

What is the value of k at the imaginary axis?

$$1 + \frac{k}{s(s+4)(s+5)}$$

$$s^3 + 9s^2 + 20s + k$$

s^3	1	20
s^2	9	k
s^1	$\frac{9\times 20 - k}{9}$	
s^0	k	

$0 < k < 180$

at $k = 180$ row $s^1 = 0$

$\rightarrow 9s^2 + k = 0$ $\quad k = 180$

$s^2 = 20$

$\boxed{s = \pm 2\sqrt{5}}$

use row above.

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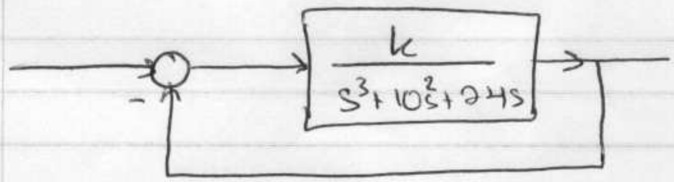
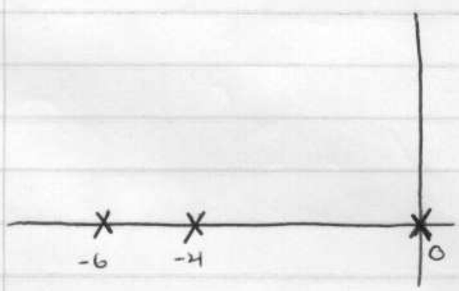
The root-locus is a plot of the roots of the characteristic equation of the closed loop system as a function of the Gain.

<u>example</u>



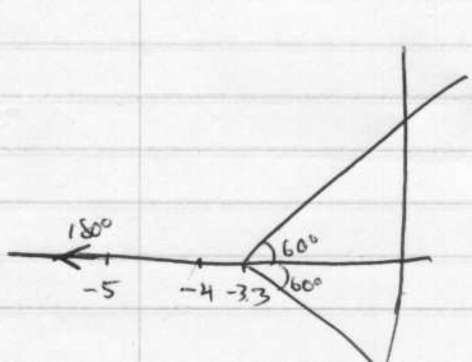
① Factor: $\frac{1}{s^3+10s^2+24s} = \frac{1}{s(s+4)(s+6)}$

② Plot poles and zeros



③ Asymptotes: $\angle s = \frac{180 + k360}{\underset{n-m}{n-d}} = \frac{180 + k360}{3 - 0} = 60, 180, -60$

$\sigma_c = \frac{\Sigma p - \Sigma z}{n-m} = \frac{0 + (-4) + (-6) + 0}{3} = \frac{-10}{3} = -3\tfrac{1}{3}$



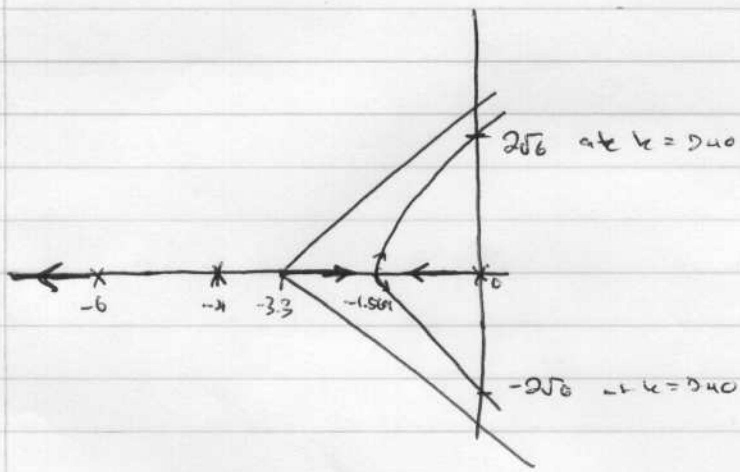
④ Real Axis

④ Breakaway Point. $S(s+4)(s+6)+k=0 = s^3+10s^2+24s+k=0$

$\rightarrow k = -(s^3+10s^2+24s)$

$\frac{dk}{ds} = -(3s^2+20s+24) = 0$

$s = \frac{-20 \pm \sqrt{400-24\cdot 3\cdot 4}}{6} = \underline{-1.569}, \ -5.096$



⑤ Imaginary Axis

s^3	1	24
s^2	10	k
s^1	$\frac{240-k}{-10}$	0
s^0	k	

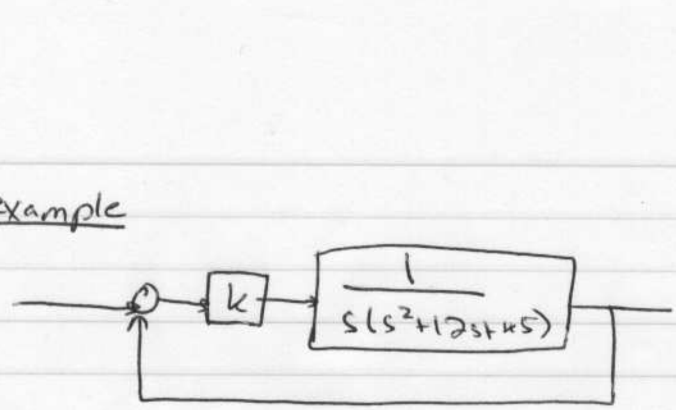
$\frac{240-k}{-10} > 0$ $\quad 240 > k$

at $k = 240$, s^1 line becomes 0,
so go one line above.

$10s^2 + k = 0 \qquad s^2 = \frac{-240}{10} = j2\sqrt{6}$

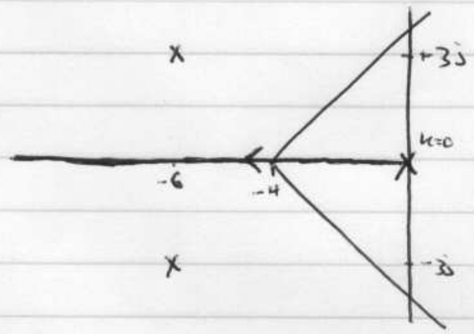
Matlab: Rlocus

example



$s^2 + 12s + 45 = 0$

$S = -6 \pm 3j, 0$



Asymptotes: $\angle S = \dfrac{180 + k360}{d - n = 3} = \begin{matrix}60\\180\\-60\end{matrix}$

$\sigma_c = \dfrac{\sum P - \sum Z}{d - n} = \dfrac{-6+3j + -6-3j}{3} = -4$

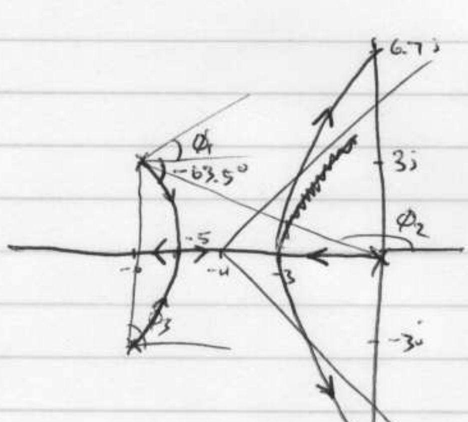
$T.F. = \dfrac{k}{s(s^2+12s+45) + k} = \dfrac{k}{s^3 + 12s^2 + 45s + k}$

s^3	1	45
s^2	12	k
s^1	b	0
s^0	k	

$b = \dfrac{(12 \cdot 45) - k}{-12} > 0 \quad k < 12 \cdot 45 \quad \underline{\underline{k < 540}}$

use S^2 line $\quad 12s^2 + k = 0 \rightarrow s = \sqrt{\dfrac{-540}{12}}$

$s = \pm j6.7$



$\underset{\text{poles}}{\sum \phi_i} - \underset{\text{zeros}}{\sum \psi_i} = 180°$ Angle of departure

$\phi_1 + (180 - 26.5°) + 90° = 180$

$\underline{\phi_1 = -63.5°}$

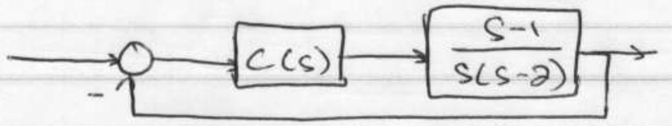
Breakaway Point. $\dfrac{dk}{ds} = 0$

$k = -(s^3 + 12s^2 + 45s) \quad = 0 = -(3s^2 + 24s + 45)$

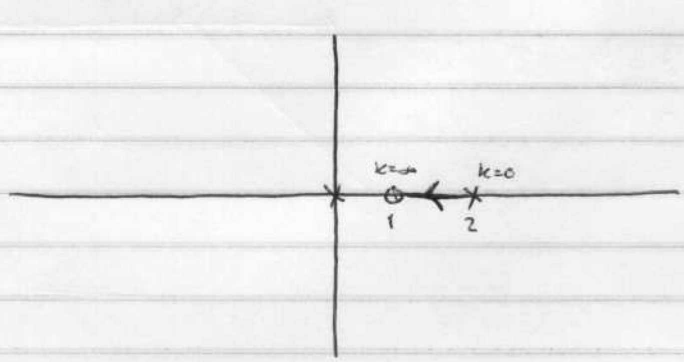
$S = -3, -5$

$k = 54, 50$

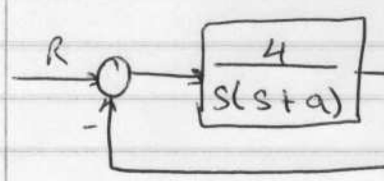
Hilroy



Check if the closed loop system can become stable for any stable $C(s)$.



no matter what $C(s)$ is, there will always be at least one pole on the RHS, so it will always be unstable.



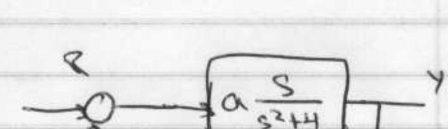
now we have a pole that changes.

$$\frac{Y}{R} = \frac{4}{s^2+as+4}$$

$$s^2+as+4=0$$

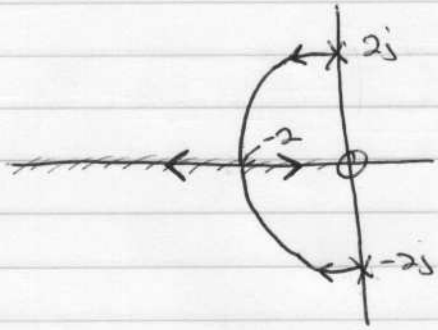
$$(s^2+4)+as=0$$

$$1+\frac{as}{s^2+4}=0 \quad \rightarrow$$



Zero : $s=0$

Pole : $\pm 2j$



angle of departure from 2j

$\sum \phi_i - \sum \psi_i = 180$

$90 + \phi_1 - 90 = 180$

$\phi_1 = 180°$

Breaking point.

$\frac{dk}{ds} = 0$

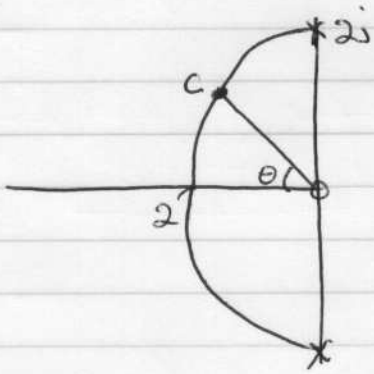
$s^2 + as + 4 = 0$

$a = \frac{-4 - s^2}{s}$

$\frac{da}{ds} = \frac{-2s^2 - (-4 - s^2)}{s^2} = \frac{s^2+4}{s^2} = 0$

$\underline{\underline{s = -2}}$

for P.O.



$\theta = \cos^{-1} \zeta$
$= 45°$

$GH = \frac{-1}{a} = \frac{s}{s^2+4}$

-> find location of c and sub in for s.

$a = \frac{1}{\frac{2}{1.33} + \frac{1}{1.37}} = \underline{2.83}$

Limitations of Root Locus

-> can only give information about poles, not zeroes.
-> can design a single parameter.
-> effictive for low order systems.

$$R(s) \xrightarrow{e^{j\omega t}} \boxed{G(s)} \longrightarrow Y(s) \qquad Y(s) = R(s)G(s)$$

$$e^{\alpha t} = \frac{1}{s-\alpha} \rightarrow Y(s) = \frac{1}{s-j\omega}G(s)$$

Assume $G(s)$ is stable

$$G(s) = \frac{q(s)}{(s-P_1)\dots(s-P_d)} \quad \rightarrow P_d \text{ is } -ve$$

$$Y(s) = \frac{q(s)}{(s-P_1)\dots(s-P_d)(s-j\omega)} = \frac{A}{s-j\omega} + \frac{B_1}{s-P_1} + \dots + \frac{B_d}{s-P_d}$$

$$A = \left.\frac{q(s)}{(s-P_1)\dots(s-P_d)}\right|_{s=j\omega} = G(s)\Big|_{s=j\omega} = G(j\omega)$$

$$y(t) = G(j\omega)e^{j\omega t} + B_1 e^{P_1 t} + \dots + B_d e^{P_d t}$$

$$\lim_{t\to\infty} y(t) = G(j\omega)e^{j\omega t}$$

$$\xrightarrow{\sin\omega t} \boxed{G(s)} \longrightarrow$$

$$\xrightarrow{\cos\omega t + j\sin\omega t} \boxed{G(s)} \longrightarrow G(j\omega)e^{j\omega t} \qquad G(j\omega) = Ae^{j\psi}$$

$$Ae^{j(\omega t+\psi)}$$

$$A\cos(\omega t+\psi) + jA\sin(\omega t+\psi)$$

$$\cos \omega t \longrightarrow \boxed{G(s)} \longrightarrow A\cos(\omega t + \varphi)$$

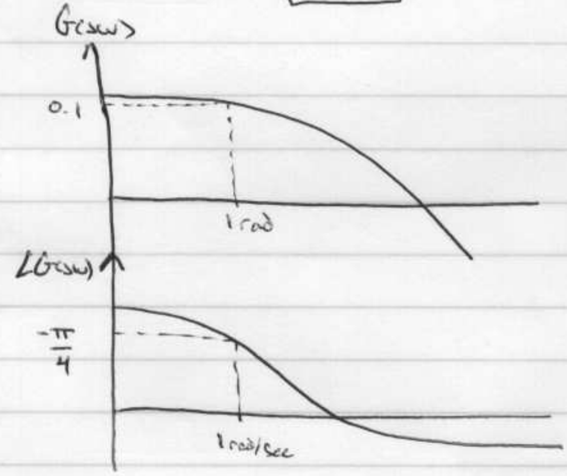
$$\sin \omega t \longrightarrow \boxed{G(s)} \longrightarrow A\sin(\omega t + \varphi)$$

$$\sin(\omega t + \phi) \longrightarrow \boxed{G(s)} \longrightarrow A\sin(\omega t + \varphi + \phi)$$

$G(j\omega) = |G(j\omega)| \angle G(j\omega)$ Plotting this is a "Bode Plot"

ex)

$$2\sin(t + \tfrac{\pi}{4}) \longrightarrow \boxed{G(s)} \longrightarrow Y(s)$$



What is $Y(t)$?

$(2)(0.1)\sin(\omega t - \pi/4) \quad \rightarrow \quad t + \pi/4 - \pi/4 = t$

$$\boxed{Y(t) = 0.2\sin t}$$

Bode Plot of 2nd Order Transfer Functions

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2/\omega_n^2}{(s/\omega_n)^2 + (2\zeta/\omega_n)s + 1} \qquad 0 < \zeta < 1$$

$$G(j\omega) = \frac{\omega_n^2/\omega_n^2}{(j\omega/\omega_n)^2 + 2\zeta/\omega_n\, j\omega + 1} = \frac{1}{(1 - \omega^2/\omega_n^2) + (2\zeta/\omega_n \cdot \omega) j}$$

$$|G(j\omega)| = \frac{|num|}{|den|} = \frac{1}{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]^{1/2}}$$

$$\angle G(j\omega) = -\tan^{-1} \frac{\frac{2\zeta\omega}{\omega_n}}{1 - \frac{\omega^2}{\omega_n^2}}$$

θ, $\frac{2\zeta\omega}{\omega_n}$, $1 - \omega^2/\omega_n^2$

$$= \angle num - \angle den$$

$$M(\omega) = 20 \log |G(j\omega)| = +\frac{20}{2} \log \frac{1}{\left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]}$$

$$= -10 \log \left[\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{2\zeta\omega}{\omega_n}\right)^2\right]$$

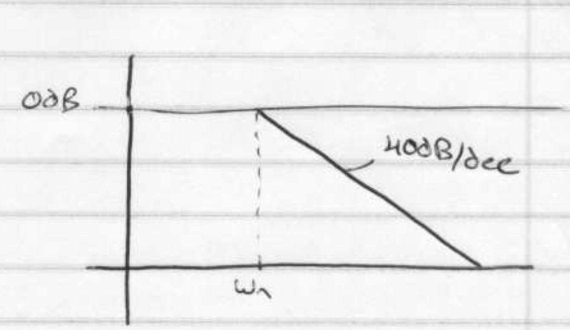
assume $\omega \ll \omega_n \rightarrow \omega/\omega_n \ll 1$

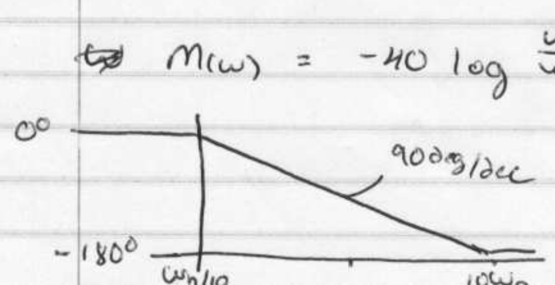
$$M(\omega) = -10 \log (1) \approx 0 \text{ dB}.$$

assume $\omega \gg \omega_n \rightarrow \omega/\omega_n \gg 1$

$$M(\omega) \approx -10 \log (\omega/\omega_n)^4$$

$$M(\omega) = -40 \log \frac{\omega}{\omega_n}$$



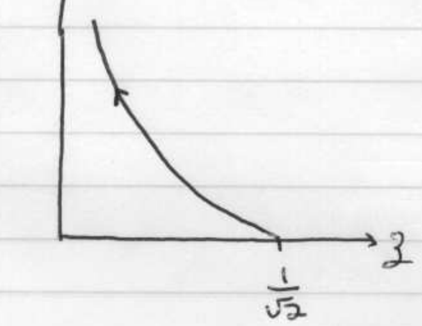


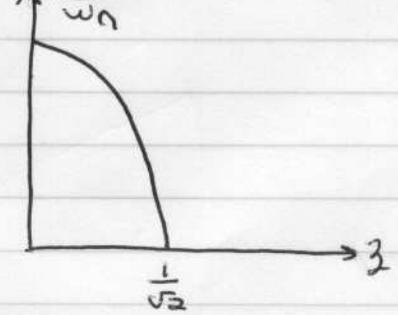
Resonance Frequency, ω_r

$M(\omega)$ is a maximum at ω_r

$\frac{d M(\omega)}{d\omega} = 0 \qquad \omega_r = \omega_n\sqrt{1-2\zeta^2} \qquad 0 \le \zeta \le \frac{1}{\sqrt{2}} \approx 0.707$

$M_r = M(j\omega_r) = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$



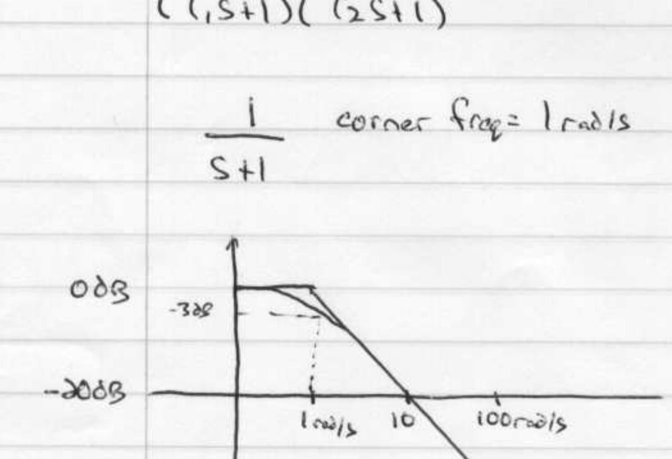


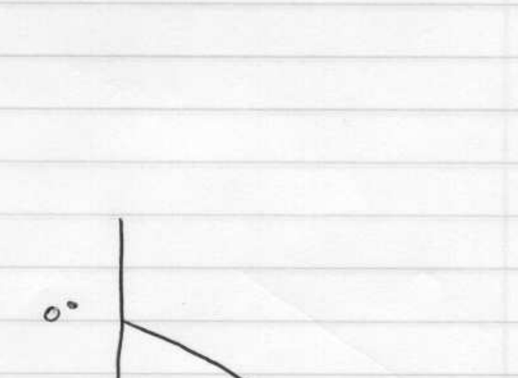
example

$\dfrac{100}{(s+1)(s/100+1)}$ Find: $G(j\omega)$, $20\log|G(j\omega)|$, and $\angle G(j\omega)$

$= \dfrac{k}{(\tau_1 s+1)(\tau_2 s+1)}$ $\tau_1 = 1$; $\tau_2 = 1/100$, $k = 100$

$\dfrac{1}{s+1}$ corner freq = 1 rad/s





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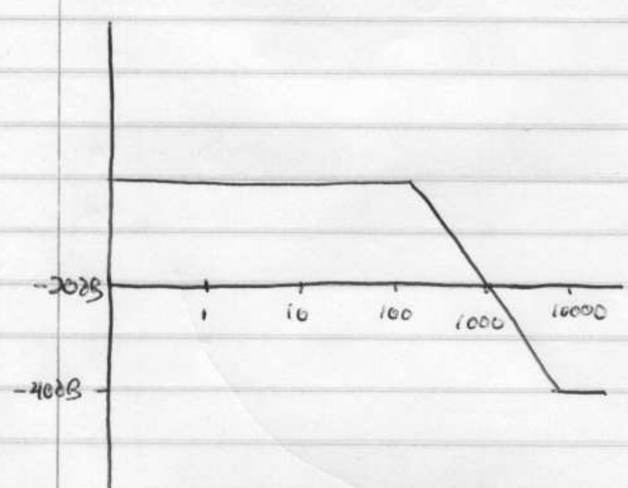
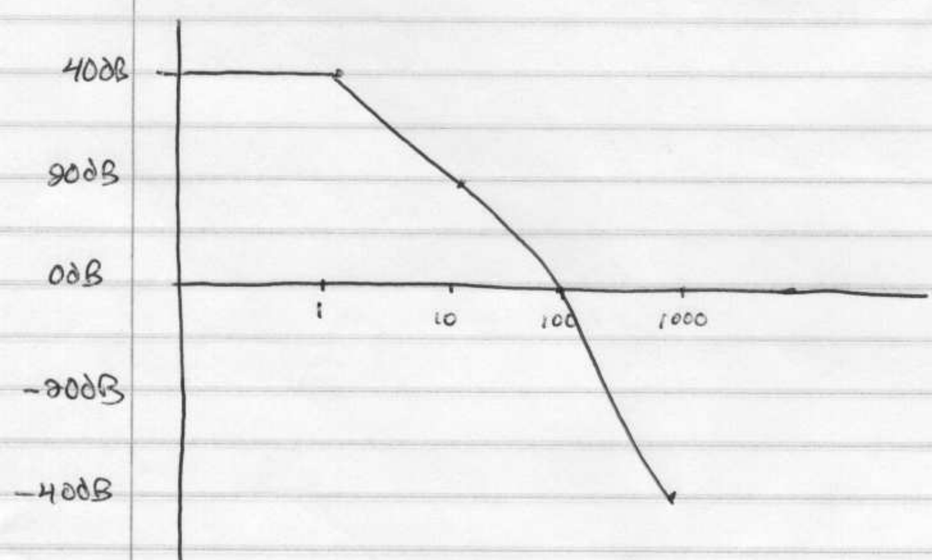
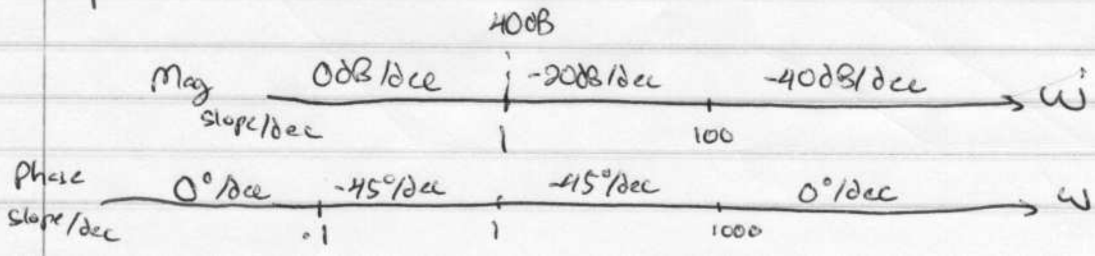
$\dfrac{1}{\frac{s}{100}+1}$ corner freq = 100 rad/sec



$K=100 \Rightarrow 20\log 100 = 40dB$

Superimpose the graphs





<u>example</u>

$$G(s) = \frac{40s(s+2)}{(s+5)(s^2+4s+100)}$$ Sketch the mag. bode plot

corner frequency : $\omega = 2$, 5 rad/s, 10 rad/s

$$G(s) = \frac{4}{25}\,\frac{s\left(\frac{s}{2}+1\right)}{\left(\frac{s}{5}+1\right)\left(\left(\frac{s}{10}\right)^2 + \left(\frac{4}{100}\right)s + 1\right)}$$

dB/dec	20	2	40	5	20	10	-20	$\to \omega$

Determine a point at low frequency

$$G(j\omega) = \frac{4}{25}\,\omega \;;\; G(j1) = \frac{4}{25} \;;\; 20\log\left(\frac{4}{25}\right) = -16\text{dB}$$

Matlab sys = G(s) -> need to input num, den

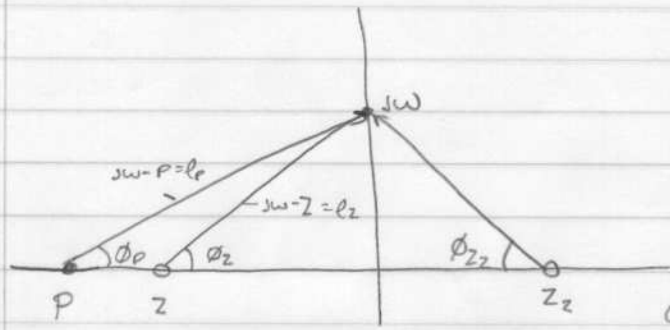
```
w = logspace (-1, 2, 1000)
[mag, ph] = bode (sys, w)
dB = 20* log10(mag)
semilogx (w, dB(1,:))
```

Can you determine the T.F / sys from the mag Bode plot?

$$G_1 = \frac{s-z}{s-p} \qquad G_1(j\omega) = \frac{j\omega - z}{j\omega - p}$$

No!!

$$G_2 = \frac{s+z}{s-p}$$

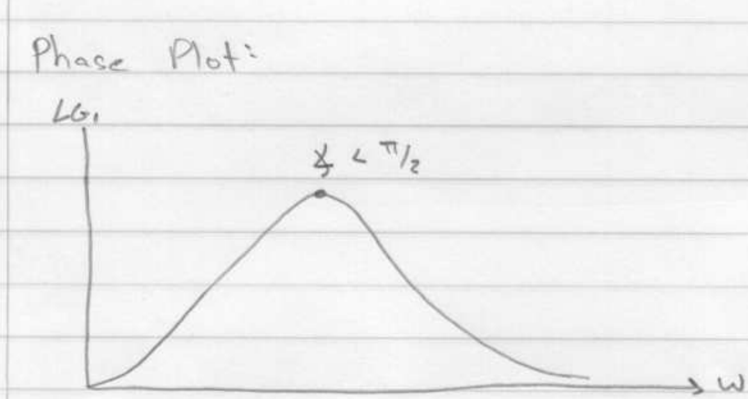


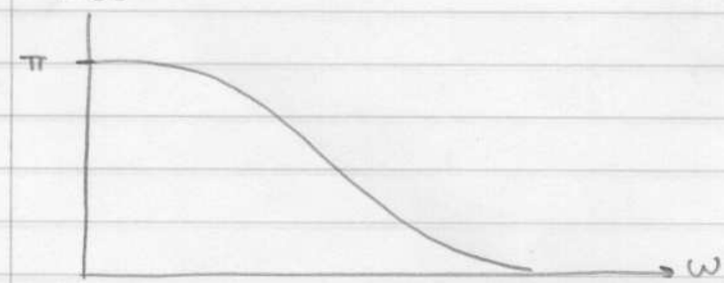
$$G_1(j\omega) = \frac{l_z \angle \phi_z}{l_p \angle \phi_p} = \frac{l_z}{l_p} \angle \phi_z - \phi_p$$

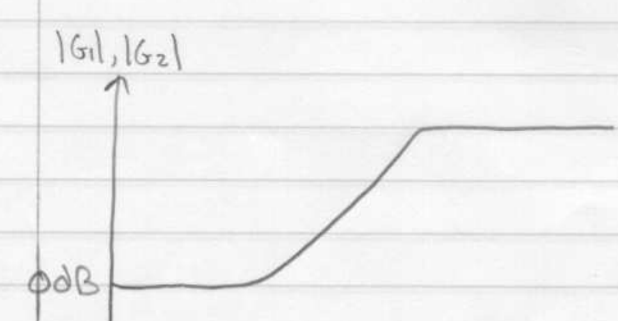
$$G_2(j\omega) = \frac{l_z}{l_p} \angle \pi - \phi_{z_2} - \phi_p$$

The magnitudes of G_1 and G_2 are the same !!!

Phase Plot:

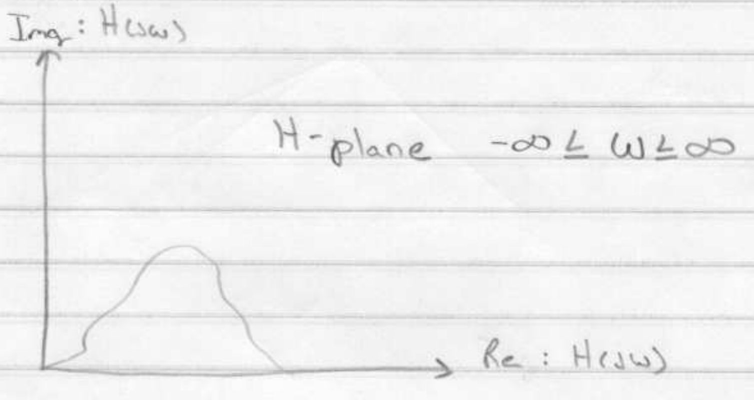


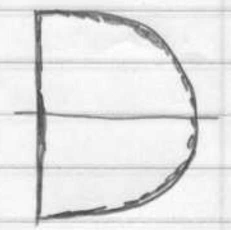




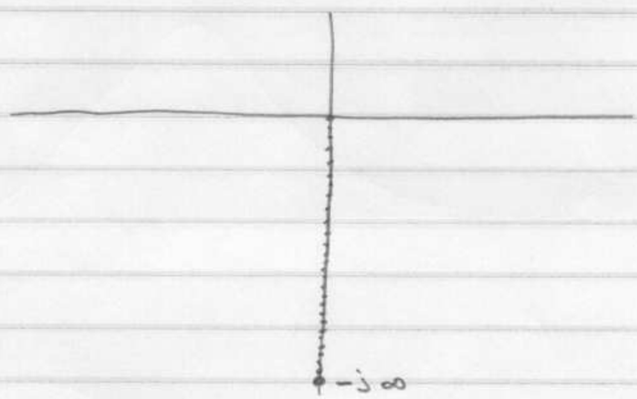
Polar Plots

S-plane - how ω changes



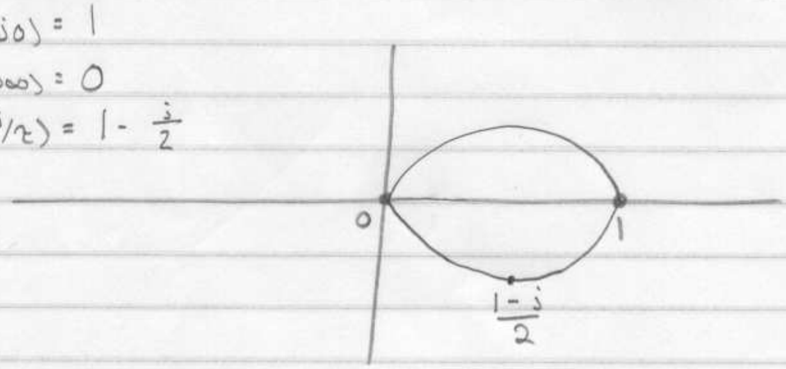


$G(s) = \frac{1}{s} \qquad G(j\omega) = \frac{1}{j\omega} = \frac{-j}{\omega}$

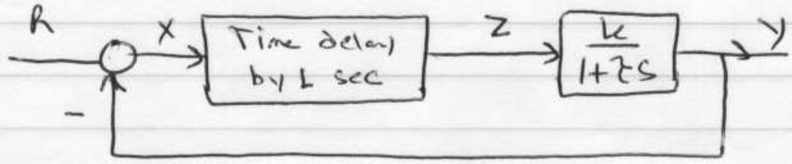


$G(s) = \frac{1}{\tau s + 1} \quad ; \quad G(j\omega) = \frac{1}{j\tau\omega + 1} = \frac{1 - j\tau\omega}{1 + \tau^2\omega^2}$

$G(j0) = 1$
$G(j\infty) = 0$
$G(j/\tau) = 1 - \frac{j}{2}$



First Order Systems with Time Delay

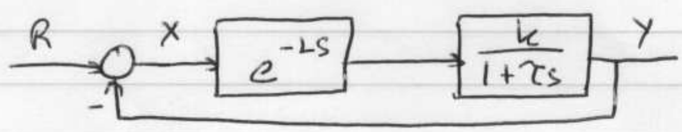


$z(t) = x(t-L)$

$x(t-L) \xrightarrow{\mathcal{L}} e^{-Ls} X(s)$

$Z(s) = e^{-Ls} X(s)$

$\frac{Z(s)}{X(s)} = e^{-Ls}$



$$\frac{Y}{R} = \frac{e^{-Ls}\frac{k}{1+\tau s}}{1 + e^{-Ls}\frac{k}{1+\tau s}} = \frac{ke^{-Ls}}{1+\tau s + ke^{-Ls}}$$

The characteristic equation is not a polynomial, so we can not use Routh-Hurwitz for stability.

Cant use Root Locus either.

Use Nyquist

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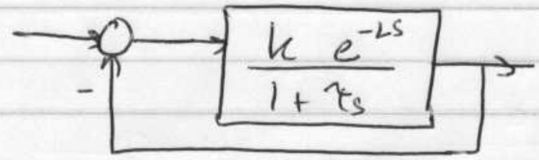
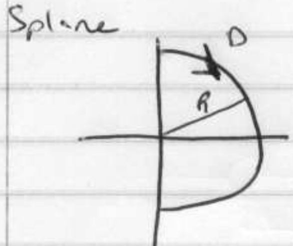


First take $k=1$

$$G(s) = \frac{e^{-Ls}}{1+\tau s}$$

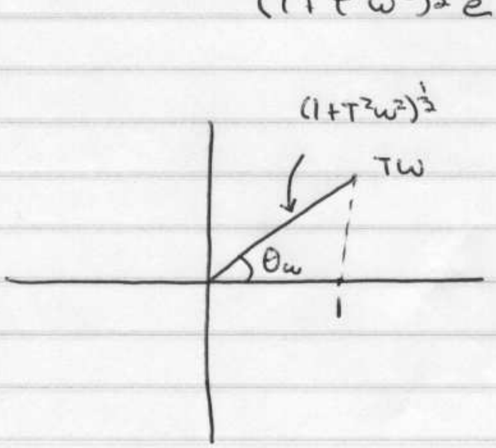
Sketch Polar Plot

Evaluate $G(s)$ along the contour D



$$G(s)\Big|_{s=j\omega} = \frac{e^{-Lj\omega}}{1+\tau j\omega} \qquad 0 \le \omega < \infty$$

$$= \frac{e^{-Lj\omega}}{(1+\tau^2\omega^2)^{\frac{1}{2}}\, e^{j(\tan^{-1}(\tau\omega))}} = \frac{1}{(1+T^2\omega^2)^{\frac{1}{2}}}\, e^{-j(L\omega + \tan^{-1}(T\omega))}$$

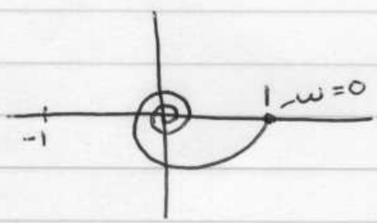


$$\theta_\omega = \tan^{-1}(T\omega)$$

$|G(j\omega)| = \frac{1}{(1+T^2\omega^2)^{\frac{1}{2}}}$ $\quad \angle G(j\omega) = -(L\omega + \tan^{-1}(T\omega))$

$G(j\omega)$: $\underset{\omega=0}{1} \to 0$ as $\omega \to \infty$

$\angle G(j\omega)$: keeps increasing -vely

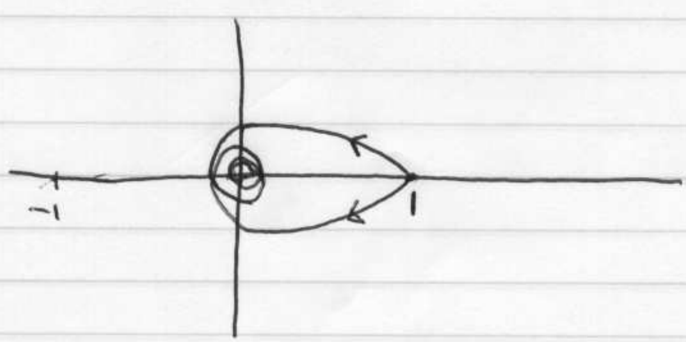


Now: $G(s)\Big|_{s=Re^{j\theta}}$ $\quad R \gg 1 \quad \theta$ is between $\frac{\pi}{2}$ and $\frac{-\pi}{2}$

$G(Re^{j\theta}) = \frac{e^{-LRe^{j\theta}}}{1+TRe^{j\theta}} = \frac{e^{-LRe^{j\theta}}}{TRe^{j\theta}} = 0$ b/c $R \gg 1$

The large semi-circle is mapped to the origin.

$G(s)\Big|_{s=j\omega} = \quad -\infty < \omega \le 0$



Where are the points of intersection?

at $\phi = \pi$

$Lw + \tan^{-1} Tw = (2k-1)\pi$ $k \to$ which intersection point

at the first point, $k=1$

$T = 5$ sec
$L = 3.2$ sec

$Lw_n + \tan^{-1} Tw_o = \pi$

$w_n = \dfrac{\pi - \tan^{-1} Tw_{n-1}}{L}$ $\to$ iterations

$w_n = -0.3148$

The open loop system has a pole $1 + TS = 0$: $S = -\frac{1}{T}$.

$-\frac{1}{T}$ is in the LHS plane.

Therefore the number of poles of the O.L. system is equal to 0 in the RHS plane $P = 0$.

$N = Z - P$ $Z = N + P$ $Z =$ number of roots of the characteristic equation.

$0.3148\,k > 1$
The system will not be stable for $\boxed{k > 3.127}$

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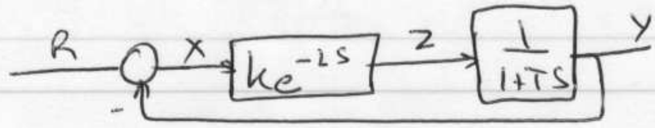


$$\frac{Y}{Z} = \frac{1}{1+TS} \qquad (1+TS)Y = Z \qquad Y(t) + T\frac{dy}{dt} = Z(t) \quad ③$$

$$Z(t) = kx(t-L) \quad ①$$

$$X(t) = r(t) - y(t) \quad ②$$

$$T\frac{dy}{dt} + y(t) = k\left(r(t-L) - y(t-L)\right)$$

$$\frac{dy}{dt} = \frac{y\{(n+1)\Delta\} - y(n\Delta)}{\Delta}$$

$$\frac{T}{\Delta}\left[y[(n+1)\Delta] - y(n\Delta)\right] + y(n\Delta) = k\left[r(n\Delta - L) - y(n\Delta - L)\right]$$

Take $\Delta = kL$ and solve using iterations.

Relative Stability

K is not actually present, we are adding it to measure the stability.

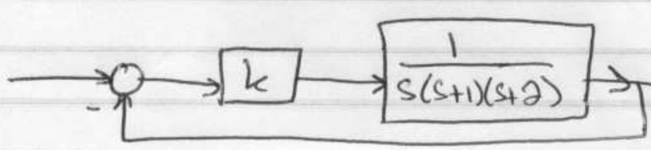
As k increases, a system can become unstable. So we want to find the maximum value of k so that the system is still relatively stable, or marginally stable.

- Assume that the C.L. system is stable

- The largest real # k, denoted k_{max}, such that the C.L. system is stable for $1 \le k \le k_{max}$

$$GM = 20 \log k_{max} \qquad \text{gain margin.}$$

example



$$T.F. = \frac{k}{S(S+1)(S+2)+k} = \frac{k}{S^3+3S^2+2S+k}$$

$$\begin{array}{c|cc} S^3 & 1 & 2 \\ S^2 & 3 & k \\ S^1 & \frac{6-k}{k} & \\ S^0 & k & \end{array}$$

$\frac{6-k}{k} > 0 \Rightarrow 0 < k < 6$

$GM = 20 \log 6 = \underline{15.56\,dB}$

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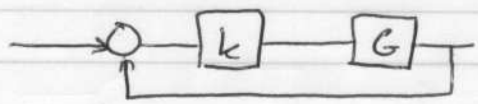
Another method: Gain Margin



$PC = G$

$1 + kG = 0$

$1 + kG(j\omega_{pc}) = 0$ phase crossover frequency

$G(j\omega_{pc}) = \frac{-1}{k_{max}}$; $k_{max} = \frac{-1}{G(j\omega_{pc})}$

$\boxed{\angle G(j\omega_{pc}) = \pm 180^\circ}$

$k = \frac{1}{|G(j\omega_{pc})|} = k_{max}$; $G_m = -20 \log |G(j\omega_{pc})|$

To find ω_{pc}, draw bode plot of O.L. system.
Locate 180°, to find ω_{pc}, then go to magnitude plot.

The gain margin (Gm) is positive for a stable system.

or Nyquist

Find intersecting point on polar plot. P_i
A system is marginally stable for an intersecting point of -1, so

$k_{max} \cdot P_i = -1$
point of intersection of real axis

$k_{max} |G(j\omega_{pc})| = 1$

Or phase Method

Phase Margin - assume C.L. system is stable

The phase margine is the largest real ϕ, ϕ_{max}, such that the C.L. system is stable for $0 \le \phi \le \phi_{max}$

The unit is in degrees.

$1 + e^{-j\phi} G(s) = 0$

$1 + e^{-j\phi}(G(j\omega_{gc})) = 0$ gain crossover frequency

$G(j\omega_{gc}) = \frac{1}{e^{-j\phi}} = e^{j\phi}$

$$\boxed{|G(j\omega_{gc})| = 1}$$

= phase $= 180° + \angle G(j\omega_{gc}) = \phi_{max}$

Find point where gain is 1, basically 0dB line, to find the frequency. Then go to phase plot.

for previous example

$\omega_{gc} = 0.4$
$\phi_{max} = 180 + (-150°) = 30° = PM$ phase margin

With nyquist, draw circle of radius 1, find angle between real axis and point where polar plot crosses the circle with radius = 1

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Phase Margin of a Second Order System

$$G = \frac{\omega_n^2}{s(s + 2\zeta\omega_n)}$$

$$\left| \frac{\omega_n^2}{j\omega_{gc}(j\omega_{gc} + 2\zeta\omega_n)} \right| = 1 = \left| \frac{\omega_n^2}{\omega_{gc}^2 + j2\zeta\omega_n\omega_{gc}} \right|$$

$$\omega_n^2 = \left((\omega_{gc}^2)^2 + (j2\zeta\omega_n\omega_{gc})^2 \right)^{\frac{1}{2}}$$

$PM = 100\zeta$ PM in degrees

We want PM at least 30°, but usually closer to 60°.

If we increase the phase margin, the system will behave better.

How can we increase phase margin?

1) Introduce a gain k, that is less than 1. This isn't a good idea because it will increase steady state error.

2) Increase the phase of the system.

Lead Compensator

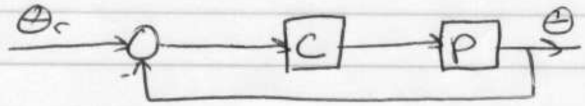
$$C(s) = \frac{\alpha \tau s + 1}{\tau s + 1} \qquad C(j\omega) = \frac{\alpha \tau j\omega + 1}{\tau j\omega + 1}$$

$$\angle C(j\omega)_{max} = \sin^{-1} \frac{\alpha - 1}{\alpha + 1} \qquad 1 < \alpha < 15 \text{ typically}$$

Thus, a lead compensator can add approximately 60°.

You want to add this phase to the ~~crossover~~ gain crossover frequency as this is where it counts the most.

Example



$$P(s) = \frac{100}{s(s+25)}$$

Specs: if Θ_r = unit ramp $e_{ss} \leq 1\%$
if Θ_r = unit step P.O. $\leq 10\%$

Question: Can we achieve the design objective using a gain controller?

spec 1) $$e_{ss} = \frac{1}{k_v} = \frac{1}{\lim_{s \to 0} sCP} = \frac{1}{\frac{100}{25} k} = 0.01$$

$$k \geq 25$$

Now find CL T.f.

$$\frac{\Theta}{\Theta_r} = \frac{100k}{s^2+25s+100k}$$

$$\omega_n = \sqrt{100k} = 10\sqrt{k} \qquad \zeta = \frac{25}{2\sqrt{100k}}$$

if we use $k \geq 25$ -> $\zeta \leq 0.25$

This results in P.O. $\geq 45\%$.

This wont Work!!

Using P.O. = 10% $\rightarrow$ need $\zeta \approx 0.6$

This results in $\phi m = 3100 \approx 60°$

Looking at O.L. bode plot, we already have 27°

We need to add 33°, but we will add 43° for some margin.

$$43° = \sin^{-1} \frac{\alpha - 1}{\alpha + 1} \quad \rightarrow \quad \alpha = 5.29$$

The new value for ω_{gc} is

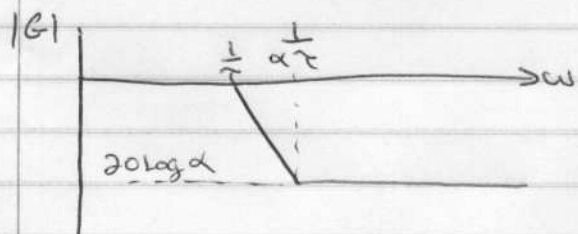
$-10 \log \alpha = 7.2dB$ this correlates to $\approx 72°$ from bode plot

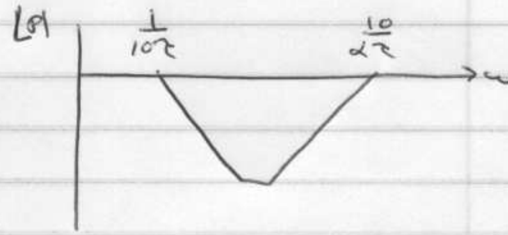
$$\omega_{max} = \frac{1}{\tau\sqrt{\alpha}} \Rightarrow 72 = \frac{1}{\tau\sqrt{5.29}} \qquad \tau = 0.006$$

Lag compensator

→ increases the PM by decreasing the gain crossover frequency (ω_{gc})

$C(s) = \frac{\alpha \tau s + 1}{\tau s + 1}$ $\qquad \tau > 0 \qquad 0 < \alpha < 1$





Using example from Nov 19th

- We needed a phase margin of 60°.
→ we will design for PM = 66°.

Looking at o.l. bode plot, we see that an attenuation of 18 dB is needed.

$20 \log \alpha = 18\ dB \qquad \alpha = 0.126$

from plot $\omega_{gc} \approx 11$ rad/s

$\omega_{gc} = \frac{10}{\alpha \tau} \qquad \tau = 6.907$

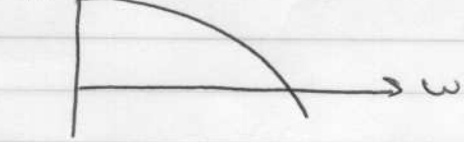
$C(s) = \frac{(6.907)(0.126)s + 1}{6.907s + 1}$

Looking at the step response of the lead and lag compensator, we find that the P.O. are similar, but the settling time is slower for the lag compensator.

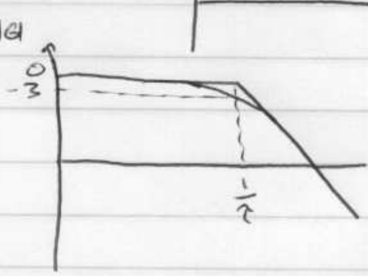
Why?

ω_{gc} increased for lead compensator, but decreased for lag compensator.

Bandwidth and Response Speed



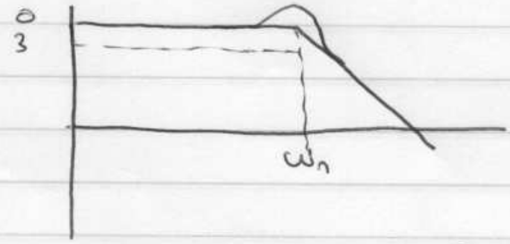
$$\frac{Y}{R} = \frac{1}{\tau s + 1}$$



$\frac{1}{\tau}$ is a corner freq. and is the bandwidth

The larger the bandwidth, the faster the system.

$$\frac{Y}{R} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$



Again, increasing bandwidth results in a faster system.

Example

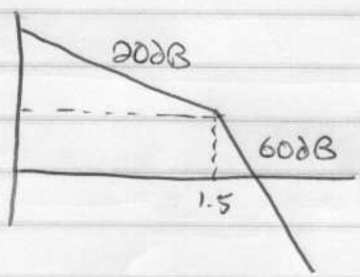
Find the gain margin of:

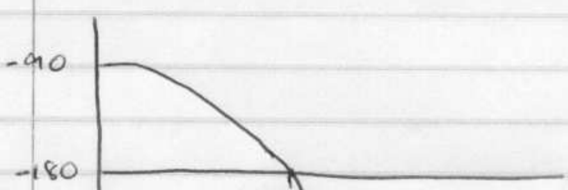
$$P(s) = \frac{2}{s(s+1.5)^2}$$

start at -90° $\quad$ $2 * 45^\circ = -90^\circ$

→ Do not find the C.L transfer function.

Plot the bode





① Find intersection of -180°, ω_{pc}

② Find $|G|$ at ω_{pc}.

$$GM = 20\log\left|\frac{1}{P(j\omega_{pc})}\right|$$
$$= -20\log|P(j\omega_{pc})|$$

$\omega_{pc} = 1.5$ rad/s

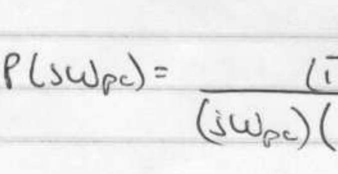
$$\boxed{GM = 10.56}$$

ω_{pc} analysis

$$P(j\omega_{pc}) = \frac{\frac{2}{(1.5)^2}}{(j\omega_{pc})\left(\frac{j\omega_{pc}}{1.5}+1\right)^2}$$

squared

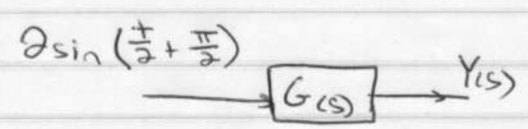
$$\rightarrow \angle P(j\omega_{pc}) = -(90^\circ + 2\tan^{-1}\frac{\omega_{pc}}{1.5}) = -180^\circ$$



$$\tan^{-1}\frac{\omega_{pc}}{1.5} = 45^\circ$$

$$\boxed{\omega_{pc} = 1.5 \text{ rad/s}}$$

Example



$G(s) = \dfrac{s-1}{s^2+3s+5}$

What is $Y(s)$?
Solve $G(s)$ at the frequency of the input.

$$G(s)\Big|_{s=j\frac{1}{2}} \quad \frac{j\frac{1}{2}-1}{(\frac{j}{2})^2+3j/2+5} = \underline{0.2245 \angle 2.37 \text{ rad}}$$

$$Y(s) = (0.2245)(2)\sin(t/2+\pi/2+2.37)$$

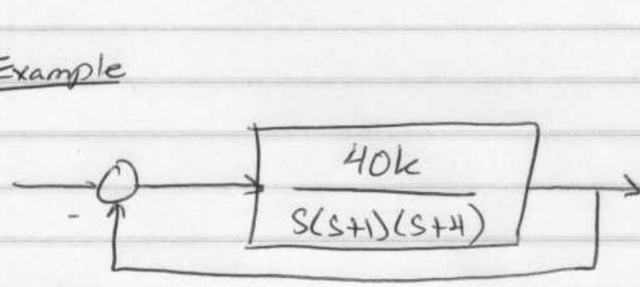
$$\boxed{Y(s) = 0.4489\sin(t/2+3)}$$

We should always check stability first!!!

If we are given the bode plot and input, look to the bode plot at $\omega = 1/2$.

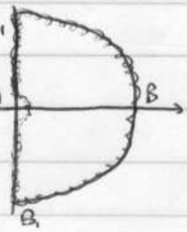
Gain $\rightarrow$ -13 $\quad\rightarrow$ $20\log G(j\omega) = -13$ $\quad G(j\omega) = \underline{0.2245}$
Phase $\rightarrow$ $136°$ $\quad\rightarrow$ $136° * \frac{\pi}{180} = \underline{2.37 \text{ radians}}$

Example



1) sketch polar plot
2) discuss stability.

S_{plane}

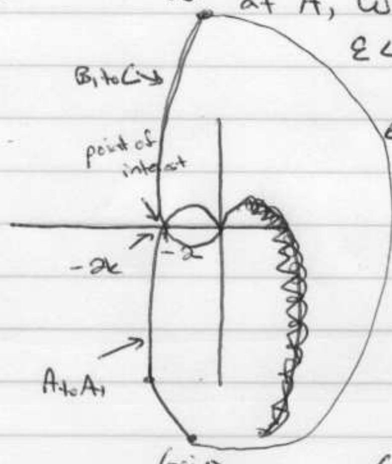


avoid O point b/c it is a pole.

ignore k for now

$$G(s)\Big|_{j\omega} = \frac{40}{S(S+1)(S+4)}\Bigg|_{j\omega}$$

at A, $\omega = \varepsilon$, $\varepsilon \ll 1$ $\quad G(j\varepsilon) = \dfrac{40}{j\varepsilon(\cancel{j\varepsilon}+1)(\cancel{j\varepsilon}+4)} = \dfrac{-j10}{\varepsilon}$



Find the value at 180° analytically:

$$G(s)\Big|_{j\omega} = \frac{40}{j\omega(j\omega+1)(j\omega+4)} = \frac{40}{j\omega(-\omega^2+5j\omega+4)}$$

$$G(j\omega) = \frac{40}{-5\omega^2 + j(4-\omega^2)\omega}$$

The point of intersection occurs when $G(j\omega)$ is a real number

so $j(4-\omega^2)\omega = 0$; $\omega = 0, \pm 2$; 0 is not a solution

$$G(j\omega)\Big|_2 \frac{40}{-5(2)^2 + \cancel{j2}} = -2$$

Now large semi-circle ; $s = Re^{j\phi}$ to find $(A, B\ B_1)$; $Re^{j\phi} \gg 1$

$$\frac{40}{Re^{j\phi}(Re^{j\phi}+\cancel{1})(Re^{j\phi}+\cancel{4})} = \frac{40}{R}e^{-j\phi 3} \rightarrow \text{all mapped to origin}$$

Now from B_1 to C, is the mirror image of A to A_1

For little semi-circle $\quad \varepsilon e^{j\phi} \quad \phi \rightarrow -\frac{\pi}{2}$ to 0 to $\frac{\pi}{2}$

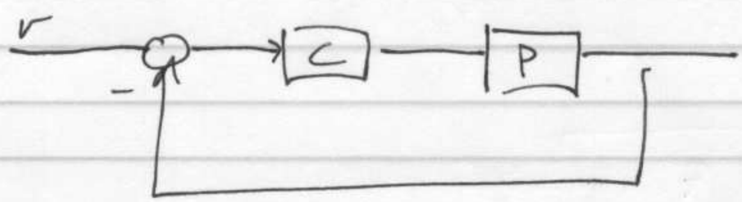
$$G(s) = \frac{40}{\varepsilon e^{j\phi}(\cancel{\varepsilon e^{j\phi}}+1)(\cancel{\varepsilon e^{j\phi}}+4)} = \frac{10}{\varepsilon} e^{-j\phi}$$

check stability.

$-2k < -1$ if $k > 0.5$ -> $N = 2$, $P = 0$ -> $Z = N + P = 2$

for stability $k < 0.5$

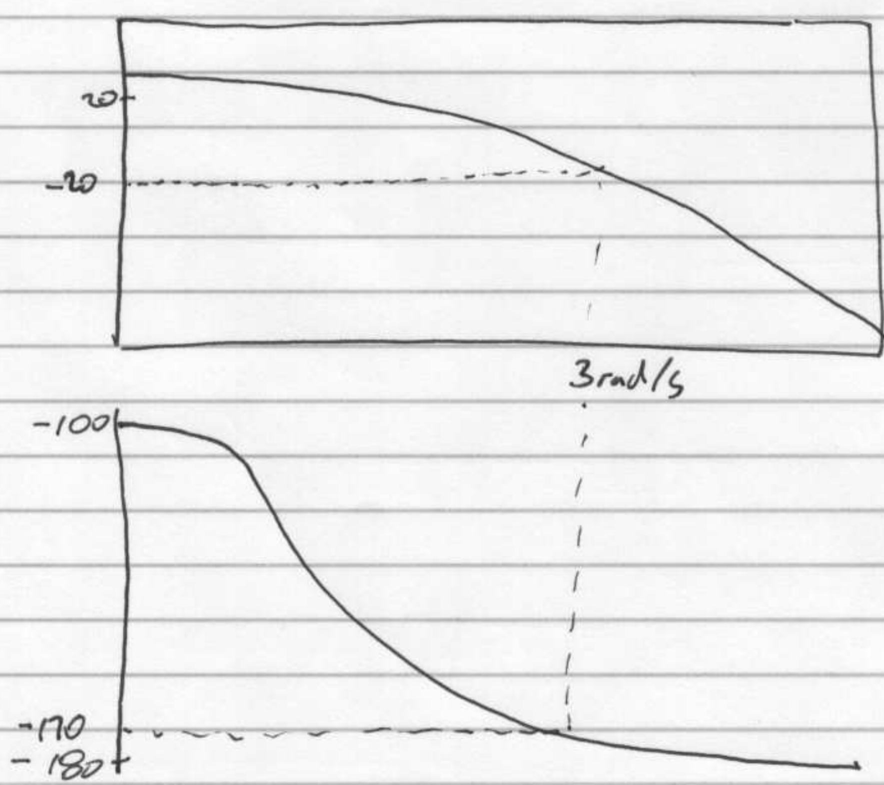
ex)



$$P(s) = \frac{2}{s(2s+1)}$$

$$C(s) = K\,\frac{\alpha \tau s + 1}{\tau s + 1} \quad \Big\}\ \text{compensator}$$

Bode Plot

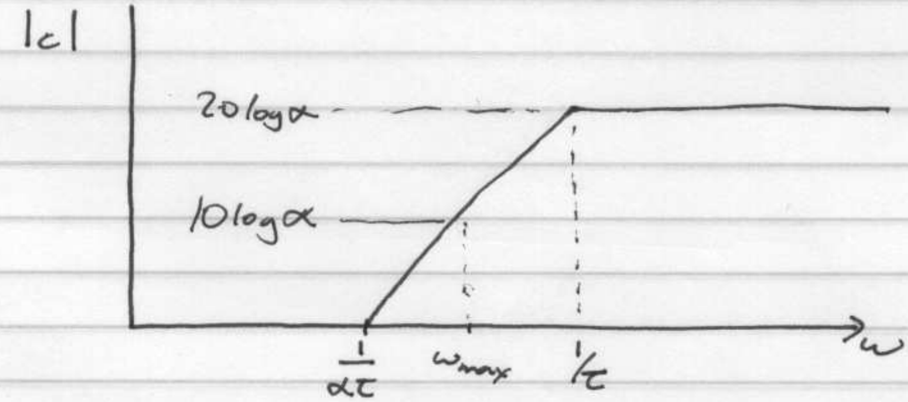


a) design a lead compensator to achieve a phase margin of 55 degrees at gain crossover of 3 rad/s

b) Design a lag compensator to achieve a PM of 45° and steady state error of 0.1 for unit ramp input

a/

$$c = (j\omega_{max}) = 10 \log \alpha$$



ω_{max} is the geometric mean of the pole and zero $\omega_{max} = \dfrac{1}{\tau\sqrt{\alpha}}$

$$\operatorname{asin}\left(\frac{\alpha-1}{\alpha+1}\right) = \measuredangle C$$

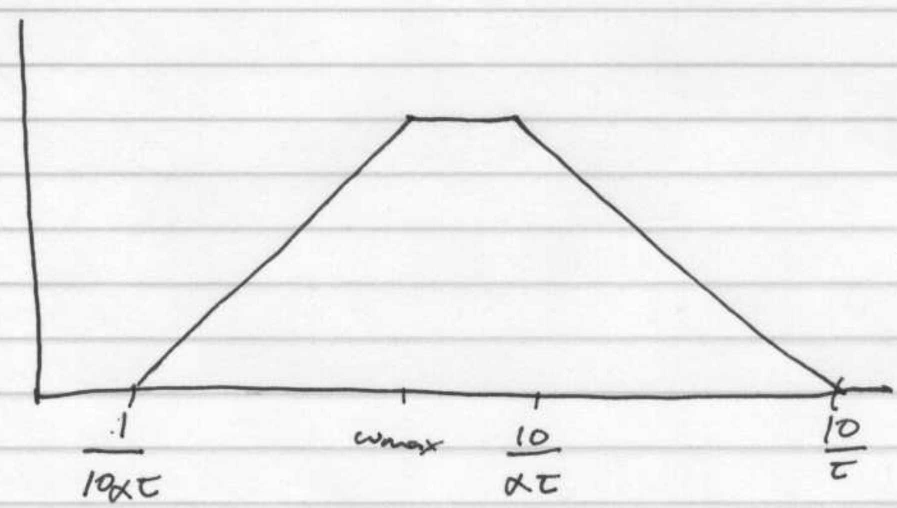
$55 - 10 = 45^\circ =$ max angle due to controller
↑ current angle at 3 rad/s

$$\operatorname{asin}\left(\frac{\alpha-1}{\alpha+1}\right) = 45$$

$$\sin(45) = \frac{\alpha-1}{\alpha+1}, \quad \alpha = 5.82$$

* see website for $\measuredangle C$ plot

$$C(s) = \frac{\alpha \tau s + 1}{\tau s + 1} \qquad \tau > 0, \quad \alpha > 1$$



$\tau = 0.1382$

choose $\omega_{gc} = \omega_{max}$ = the ω with maximum phase

really?

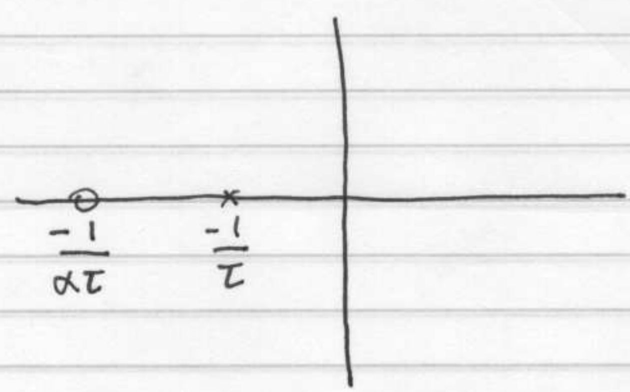
$$\underbrace{(20)|P(j\omega_{gc})| + 20\log(K) + 10\log\alpha}_{= 20\log PC(j\omega_{gc})} = 0$$

$= 20\log PC(j\omega_{gc}) = 0$

$\therefore K = 3.7$ ← correct answer

b// Design lag compensator, $PM = 45°$, $e_{ss} = 0.1$ for unit ramp input

$$C(s) = \frac{\alpha\tau s + 1}{\tau s + 1} \qquad \tau > 0, \quad 0 < \alpha < 1$$

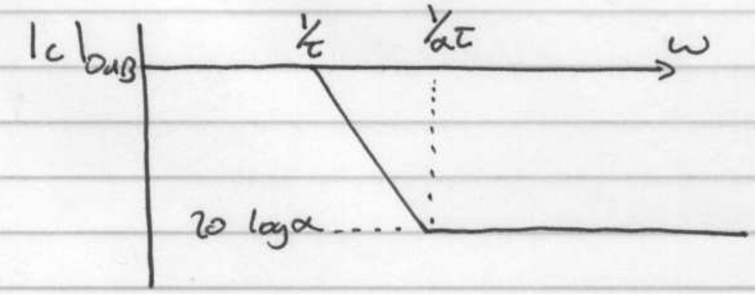


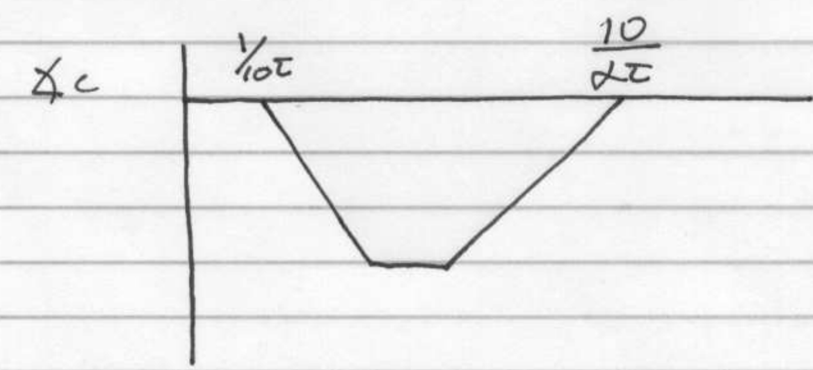
lag compensator increases the PM by decreasing the gain crossover frequency (ω_{gc})

Note: with lag compensator, you add an additional -6° to PM,

$\therefore$ make a PM $= 45 - (-6^\circ) = 51^\circ \approx 50^\circ$

$\therefore$ at phase $= 130^\circ$, make gain $= 0$





$\therefore$ at phase $= 130^\circ$, gain $= \underline{12dB}$, $\omega_{gc} = 0.4$

$$PC = \frac{2}{s(2s+1)} \; k \; \frac{\alpha\tau s+1}{\tau s+1}$$

$$K_v = \lim_{s\to 0} s\,p(s)\,c(s) = \lim_{s\to 0} \frac{\cancel{s}2}{\cancel{s}(2s+1)} \; k \; \frac{\cancel{\alpha\tau}s+1}{\cancel{\tau}s+1} = 2K$$

$$\underset{\text{(for ramp)}}{e_{ss}} = \frac{1}{K_v} = \frac{1}{2k} = 0.1 \quad , \quad \therefore k = 5$$

$$P(j\omega_{gc}) = 12\,dB$$

$$|P(j\omega_{gc})\; C(j\omega_{gc})| = 1$$

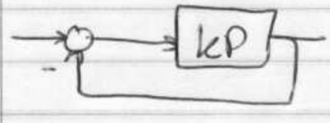
$$20 \log p(j\omega_{gc}) + 20 \log k + 20 \log \alpha = 1$$

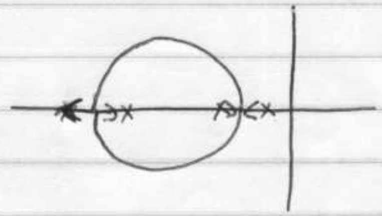
$$12 + 20 \log(5) + 20 \log(\alpha) = 1$$

$$\alpha = 0.05$$

$$\frac{10}{\alpha\tau} = \omega_{gc} = 0.4$$

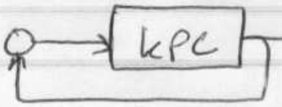
$$\tau = \frac{10}{0.4(0.05)} = 500$$





To determine if a point is on the root locus: $\measuredangle P(s) = 180^\circ$

To make a point part of the root locus, add a controller, C.



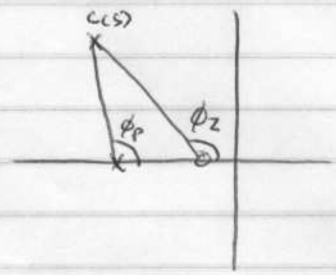
$\angle PC = 180^\circ$

$\angle P + \angle C = 180$

$$\boxed{\measuredangle C(s) = 180^\circ - \measuredangle P(s)}$$

$k\,|P(s)| = |-1|$

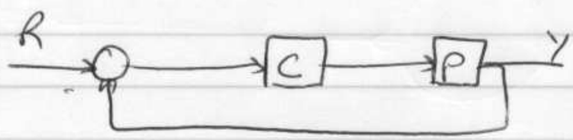
$$\boxed{k = \frac{1}{|P(s)|}}$$



$\measuredangle C(s) = \phi_z - \phi_p$

*Poles must be dominant.

<u>example</u>

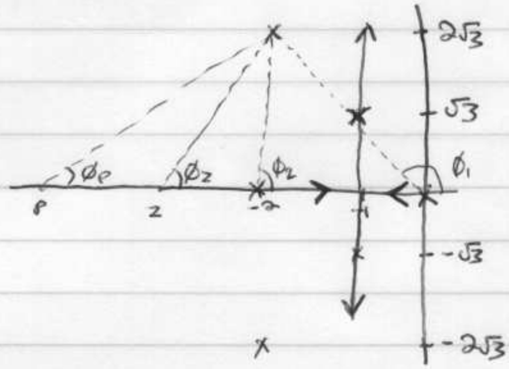


$P(s) = \frac{4}{s(s+2)}$

$C(s) = 1$

$\frac{Y}{R} = \frac{4}{s(s+2)+4}$

Char eq'n : $s^2+2s+4=0$
roots: $-1 \pm \sqrt{3}$



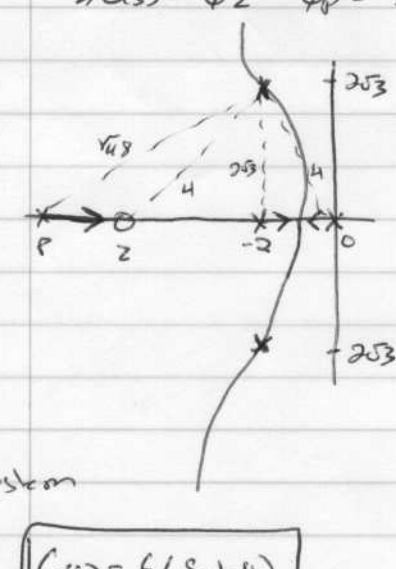
Find c(s) so that C.L. poles are at $\underline{s = -2 \pm 2\sqrt{3}}$

① Find $\measuredangle P(s)\Big|_{s=-2\pm j2\sqrt{3}} = -(\phi_1 + \phi_2) = -(120° + 90°)$
$= -(\tan^{-1}\frac{\sqrt{3}}{-1} + 90°)$
$= -210°$

$\measuredangle C(s) = 180 - \measuredangle P(s) = 180 + 210 = 390° = 30°$

$\measuredangle C(s) = \phi_z - \phi_p = 30°$

Suppose $\phi_z = 60°$, $\phi_p = 30°$



then $z = -4$, $p = -8$

$C(s) = \frac{s+4}{s+8} * k_c$ ← constant

$k_c = \frac{1}{|PC|} = \left|\frac{s(s+2)(s+8)}{4(s+4)}\right|$, $s = \left|\sqrt{2^2+(2\sqrt{3})^2}\right|$

$s = 4$
$|s+4| = 4$
$|s+2| = 2\sqrt{3}$
$|s+8| = \sqrt{6^2+(2\sqrt{3})^2} = \sqrt{48}$

$k_c = \frac{(4)(2\sqrt{3})(\sqrt{48})}{(4)(4)} = \boxed{6}$

Poles of C.L. system
$s = 6,$ ~~20~~
$s = -2 \pm 2\sqrt{3}j$

$\boxed{C(s) = \frac{6(s+4)}{(s+8)}}$

ELE 481
Nov 29/04
3/3

-> go over principles and proofs

-> understand root locus and why it works.

EE 481

Dec 1/04

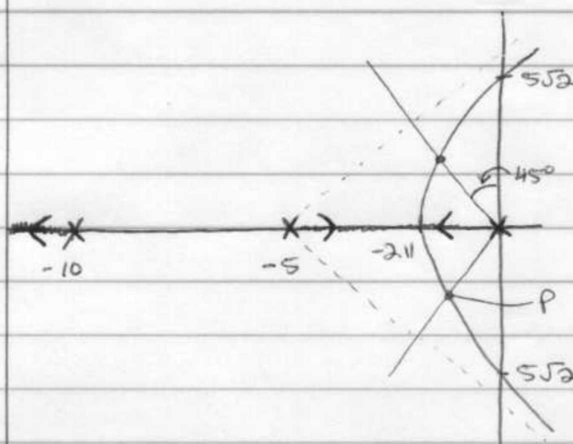
1/2

example

$$G(s) = \frac{k}{s(s+5)(s+10)} \qquad \text{given: } k_v = 1/400$$

Angles = $60°, 180°, -60°$

$$\sigma_c = \frac{-10-5-0}{3} = -5$$



Breakaway point.

$s(s+5)(s+10) + k = 0 = s^3 + 15s^2 + 50s + k = 0$

$k = -(s^3 + 15s^2 + 50s)$

$\frac{\partial k}{\partial s} = -(3s^2 + 30s + 150) = 0 \quad \Rightarrow s = -2.11$

Imaginary crossing.

s^3	1	50
s^2	15	k
s^1	$\frac{15 * 50 - k}{15}$	
s^0	k	

$k < 750$

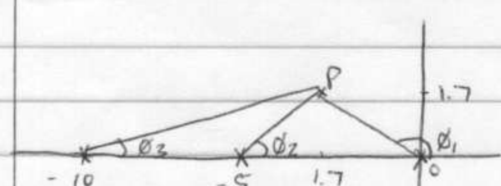
$15s^2 + k = 0 \; ; \; k = 750$

$s = \pm 5\sqrt{2}$

find k so that $\zeta = 0.7$

$\cos^{-1} 0.7 = 45°$ we want to know the location of point P.

We make a guess, say $-1.7 + j1.7$



$\angle P = -(\emptyset_1 + \emptyset_2 + \emptyset_3)$

$= (135 + 27.2 + 11)$

$= 173°$

-> b/c $\angle P \neq 180°$, this guess is not correct.

$\emptyset_1 = 180 - 45°$

$\emptyset_2 = \tan^{-1}\left(\frac{1.7}{5-1.7}\right) = 27.2°$

$\emptyset_3 = 11°$

-> An iteration at $1.9 + j1.9$, we would have found $\angle P = 180°$

point $P = 1.9 + j1.9$

now find k

$$kP(s) + 1 = 0$$

$$k = \frac{-1}{P(s)}\bigg|_{s=1.9+j1.9} \Rightarrow |k| = |s||(s+5)||(s+10)| \quad , \quad |s| = 1.9\sqrt{2}$$

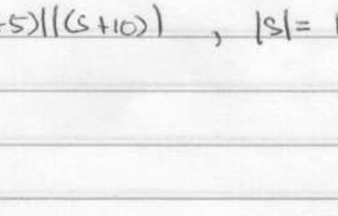
$$|s+5| = \sqrt{(5-1.9)^2 + 1.9^2} = 3.62$$
$$|s+10| = \sqrt{(10-1.9)^2 + 1.9^2} = 8.32$$

$k = 81.7$

$$k_v = \lim_{s\to 0} sG(s) = \lim \frac{sk}{s(s+5)(s+10)} = \frac{k}{50} = \frac{81.7}{50} \approx 1.6$$

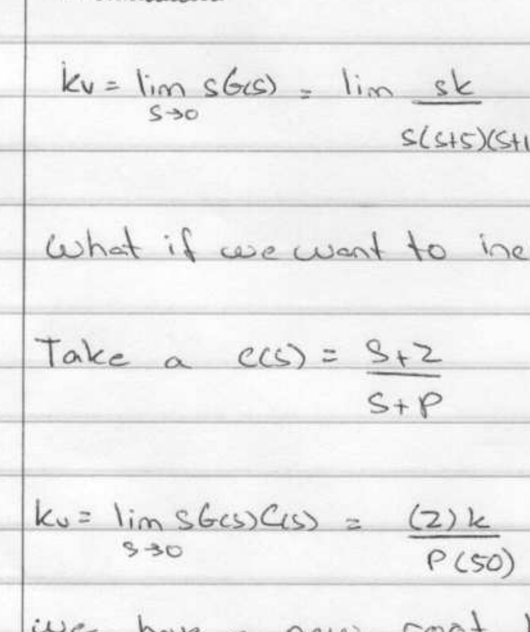
What if we want to increase k_v by a factor of 4?

Take a $C(s) = \dfrac{s+z}{s+p}$



$$k_v = \lim_{s\to 0} sG(s)C(s) = \frac{(z)k}{p(50)} \to \frac{z}{p} = 4$$

we have a new root locus with the addition of $C(s)$



choose $z = 0.2$, $p = 0.04$ } k_v will increase by a factor of 5

You must choose z and p close to the origin, approximately 10 times less than our new pole, 1.9, so we chose 0.2.

The closer to the origin you choose z and p, the slower your system will be.

State Space

If you have a system of differential equations, convert them to first order differential equations and use matricies to solve.

$$\frac{Y(s)}{R(s)} = \frac{1}{ms^2 + bs + k}$$

$$Y(s)\left[ms^2+bs+k\right] = R(s)$$

$$m\frac{\partial y^2}{\partial t} + b\frac{\partial y}{\partial t} + ky = r(t) \qquad \text{define } y = x_1$$

$$\frac{\partial y}{\partial t} = x_2$$

so, $x_2 = \dot{x}_1$

$$m\dot{x}_2 + bx_2 + kx_1 = r(t)$$

$$\dot{x}_2 = \frac{r(t)}{m} - \frac{k}{m}x_1 - \frac{b}{m}x_2$$

define $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\frac{\partial x}{\partial t} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix}}_{B} r(t)$$

$$y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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$$\begin{cases} \dfrac{\partial x}{\partial t} = Ax + Br \\ \\ Y = Cx + Du \end{cases}$$

<u>example</u>

$$\frac{Y(s)}{U(s)} = \frac{3s^2 - 2s + 1}{s^3 + 5s^2 + 4s + 2}$$

$$\frac{Y(s)}{3s^2 + 1 - 2s^2} = \frac{U(s)}{s^3 + 5s^2 + 4s + 2} = X(s)$$

$$\frac{U(s)}{s^3 + 5s^2 + 4s + 2} = X(s) \quad \rightarrow \quad U(\cancel{\$}) = X(s)\left(s^3 + 5s^2 + 4s + 2\right)$$

Inverse laplase transform $\quad X^{(3)} + 5\underset{\substack{\downarrow \\ X_3}}{X^{(2)}} + 4\underset{\substack{\downarrow \\ X_2}}{X^{(1)}} + 2\underset{\substack{\downarrow \\ X_1}}{X} = U(t)$

$X_2 = \dot{X}_1$

$X_3 = \dot{X}_2 \qquad \dot{X}_3 + 5X_3 + 4X_2 + 2X_1 = u(t)$

$$\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \\ \dot{X}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t)$$

now,

$$\frac{X(s)}{Y(s)} = \frac{1}{3s^2 + 1 - 2s}$$

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$Y(s) = 3s^2 X(s) - 2s X(s) + X(s)$

$$\underset{\dot{x}_3}{3x^{(2)}} - \underset{\dot{x}_2}{2x^{(1)}} + \underset{\dot{x}_1}{x} = y(t)$$

$$y(t) = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

General Form

$$\frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \dots + a_0}$$

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} b_0 & b_1 \cdots b_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

$\dot{x} = Ax + Bu$
$y = Cx + Du$

D will only appear if the order of num and den are equal, D may be a number you factored out.

How do we use this to study the system?

$X = T Z$ ← matrix

$Z = T^{-1} X$

$\dot{X} = T\dot{Z} = ATZ + BU \quad (1)$

$Y = CTZ + DU$

now multiply (1) by T^{-1}

$\dot{Z} = T^{-1}ATZ + T^{-1}BU$

$\dot{Z} = \bar{A}Z + \bar{B}U \qquad \bar{A} = T^{-1}AT \quad \bar{B} = T^{-1}B \quad \bar{C} = CT \quad \bar{D} = D$

$Y = \bar{C}Z + \bar{D}U$

Objective is to make $\bar{A}$ diagonal, making the system easier to study.

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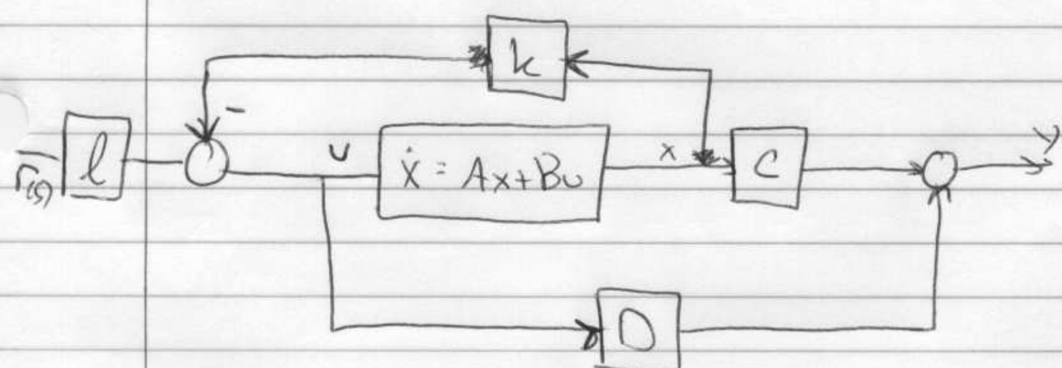
Controller Design in the State Space (pole placement)

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \frac{Y(s)}{U(s)} = \frac{b_{n-1}s^{n-1} + b_{n-2}s^{n-2} + \dots + b_0}{s^n + a_{n-1}s^{n-1} + \dots a_0}$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0\ 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 1 \end{bmatrix} u$$



$$u = rl - (k_1\ k_2 \dots k_n) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = rl - kx$$

$$\begin{cases} \dot{x} = Ax + B(rl - kx) = (A - Bk)x + lBr \\ y = cx \end{cases}$$

$$Bk = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} (k_1\ k_2 \dots k_n) = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ k_1 & k_2 & k_3 & \dots & k_n \end{pmatrix}$$

$$A - Bk = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \quad 1 \\ -a_0-k_1 & -a_1-k_2 & -a_2-k_3 & \cdots & \cdots & -a_{n-1}-k_n \end{pmatrix}$$

You can now set each elemint in the last row to any number you want and solve for the K values. You can put the poles anywhere you want on the s-plane.

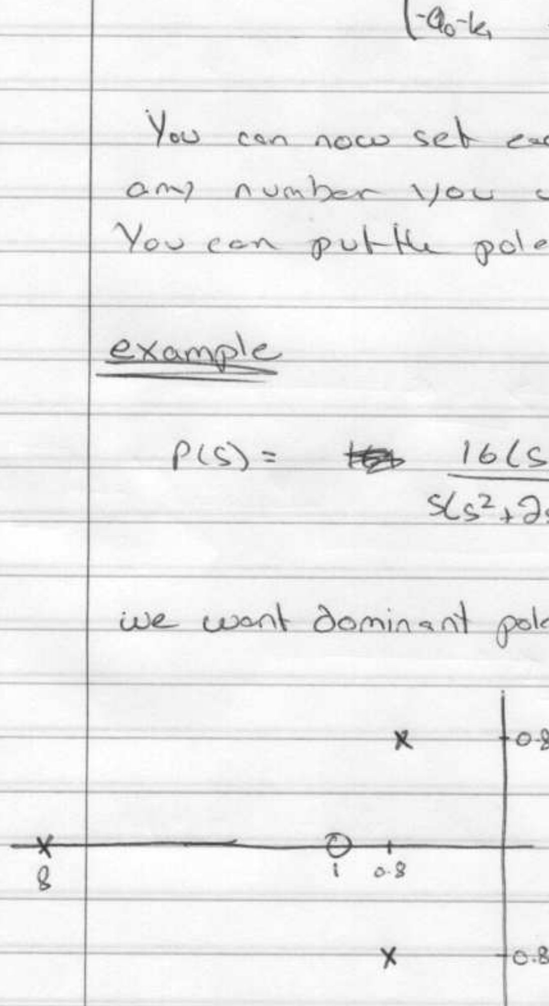
<u>example</u>

$$P(s) = \frac{16(s+1)}{s(s^2+2s+16)}$$

we want: PO = 5% $\rightarrow \zeta = 0.7$

T_s = 5 sec $\rightarrow \zeta\omega_n = 0.8$

we want dominant poles at $0.8 \pm j0.8 = S_{1,2}$



we need one more pole because it is a 3rd order system

choose, $10 \times 0.8 = 8 = S_3$

now,
$$\dot{X} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -16 & -2 \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$

$$Y = 16 \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} X$$

Desired characteristic equation

$$(s-s_1)(s-s_2)(s-s_3) = (s+0.8+j0.8)(s+0.8-j0.8)(s+8)$$

$$= \left((s+0.8)^2+0.8^2\right)(s+8) = s^3 + 9.6s^2 + 14.08s + 10.24$$

We want to obtain a transfer function of the form:

$$T(s) = \frac{10.24(s+1)}{s^3+9.6s^2+14.08s+10.24}$$

now,

$$\dot{x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10.24 & -14.08 & -9.6 \end{pmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} r$$

$$y = 10.24 \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} x$$

and,

$$\begin{array}{l|l} 0-k_1 = -10.24 & k_1 = 10.24 \\ -16-k_2 = -14.08 & k_2 = \cancel{7.6}\ -1.92 \\ -2-k_3 = -9.6 & k_3 = 7.6 \end{array}$$

Looking at step responce, we have P.O. > 5%, but our poles are where we want (root locus). Matlab on next page.

The problem is that we have a zero very close to our dominant poles.

Lets try placing the last pole at -0.9 instead of -8.

-> This works b/c the pole (-0.9) and zero (-1) cancel

Matlab code

```
a= [0 1 0; 0 0 1; 0 -16 .2];
b= [0 0 1];
P= [-0.8+0.8*j; -0.8-0.8*j; -8];
K= acker(a,b,p);
abar = a-b*k;
c =[1 1 0];
b=b';
sys1 = ss(a,b,c,[0]);
sys1t = tf(sys1)%

sys1c = ss(abar, -b*abar(3,1) , c , [0]);
sys1ct = tf (sys1c);
subplot (3,1,2);
step(sys1ct);
title('...')
subplot ( 3,1,3)
pzmap(sys1ct);
```

exam
- -> assigments
- -> exams in class
- -> fundamentals
  - -> know why a proceoure is the way its.
  - ie) why you can determine stability based on nyquist.
  - -> 75% from midterm on
- -> 1 formula sheet (2 sides)